ANALYTIC FUNCTIONS AND CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS[1]

PART I

INFINITELY DIFFERENTIABLE FUNCTIONS

1. ANALYTIC FUNCTIONS OF A REAL VARIABLE

A function $f(x)$, defined in a closed interval $I \equiv [a, b]$ is said to be *analytic* in this interval, if to every point x_0 belonging to I there corresponds a Taylor series with a positive radius of convergence, converging to $f(x)$ in a neighborhood of x_0. Since the derivative of a Taylor series is a Taylor series with the same radius of convergence as the given series, and therefore converging uniformly in every closed interval containing x_0, and contained in the interval of convergence, an analytic function is infinitely differentiable in I.

An infinitely differentiable function in $[a, b]$ will be said to be an "*i. d. function*" in $[a, b]$.

It is well known, that an i. d. function in $[a, b]$ is not necessarily analytic in this interval. For instance the function $f(x)$ defined by the equalities

$$f(x) = e^{-1/x^2}, \text{ if } x \neq 0,$$
$$f(0) = 0,$$

is i. d. in every closed interval, but is not analytic in any interval containing the point $x = 0$.

[1]A series of lectures delivered at the Rice Institute during the academic year, 1940-41, by S. Mandelbrojt, Docteur ès Sciences (Paris), Professor at the Collège de France, Visiting Professor of Mathematics at the Rice Institute.

It is also easy to construct an i. d. function in I which is not analytic in *any* partial interval of I. It would seem to be possible to construct such an example by the well known "principle of condensation of singularities" of du Bois-Raymond, but, for our purpose, it is more convenient to operate in a different manner.

Consider, for instance,

$$f(x) = \sum_{n=1}^{\infty} e^{-(n!)^{\frac{1}{2}}} \cos(n!x).$$

This function is i. d. in the interval $[0, 2\pi]$, since for every integer $p \geqq 0$, the series

$$(-1)^p \sum e^{-(n!)^{\frac{1}{2}}} (n!)^{2p} \cos(n!x)$$
$$(-1)^p \sum e^{-(n!)^{\frac{1}{2}}} (n!)^{2p+1} \sin(n!x)$$

converge uniformly in $[0, 2\pi]$ and represent respectively $f^{(2p)}(x)$ and $f^{(2p+1)}(x)$.

If l and m are two integers such that

$$0 \leqq l \leqq m,$$

then

$$\pm f^{(2p)}\left(\frac{2l\pi}{m}\right) = \sum_{n=1}^{\infty} e^{-(n!)^{\frac{1}{2}}} (n!)^{2p} \cos\left(\frac{2ln!\pi}{m}\right)$$
$$= \sum_{n=1}^{m-1} e^{-(n!)^{\frac{1}{2}}} (n!)^{2p} \cos\left(n!\frac{2l\pi}{m}\right) + \sum_{n=m}^{\infty} e^{-(n!)^{\frac{1}{2}}} (n!)^{2p}.$$

The sum extended from 1 to $m-1$, which we shall denote by $A(l, m, p)$, satisfies the inequality

$$|A(l, m, p)| < (m!)^{2p}.$$

Denoting by $B(m, p)$ the sum extended from m to infinity, we may write

$$\left(\frac{\left|f^{(2p)}\left(\frac{2l\pi}{m}\right)\right|}{(2p)!}\right)^{1/2p} \geqq \frac{\left|f^{(2p)}\left(\frac{2l\pi}{m}\right)\right|^{1/2p}}{2p} \geqq \frac{(B(m, p) - |A(l, m, p)|)^{1/2p}}{2p}$$
$$> \frac{(B(m, p) - (m!)^{2p})^{1/2p}}{2p} = \frac{m!}{2p}\left(\frac{B(m, p)}{(m!)^{2p}} - 1\right)^{1/2p}.$$

For any integer $q>m$, we may then write

$$\left(\frac{\left|f^{(2p)}\left(\frac{2l\pi}{m}\right)\right|}{(2p)!}\right)^{1/2p} > \frac{m!}{2p}\left(\frac{e^{-(q!)^{\frac{1}{2}}}(q!)^{2p}}{(m!)^{2p}}-1\right)^{1/2p}.$$

If, l and m being fixed, we choose p and q in such a manner that $2p=(q-1)!$, $q>m$, we may write

$$\left(\frac{e^{-(q!)^{\frac{1}{2}}}(q!)^{2p}}{(m!)^{2p}}\right)^{1/2p} = \frac{e^{-\frac{(q!)^{\frac{1}{2}}}{(q-1)!}}q!}{m!} = A_q,$$

and, when q tends to infinity, it is obvious that

$$\frac{m!}{2p}\left(\frac{e^{-(q!)^{\frac{1}{2}}}(q!)^{2p}}{(m!)^{2p}}-1\right)^{1/2p} \backsim \frac{m!A_q}{2p} = e^{-\frac{(q!)^{\frac{1}{2}}}{(q-1)!}}q,$$

the last quantity tending to ∞ with q.

We have thus proved that, for every point $x=\frac{2l\pi}{m}$,

$$\overline{\lim_{\nu=\infty}}\left(\frac{\left|f^{(\nu)}\left(\frac{2l\pi}{m}\right)\right|}{\nu!}\right)^{1/\nu} = \infty,$$

that is to say, the radius of convergence of the Taylor series of $f(x)$ at such a point is zero, and the function is not analytic in any interval containing these points. But every interval which is contained in $[0, 2\pi]$ contains such a point. The function $f(x)$ is therefore the desired one.

It is important to be able to distinguish an analytic function by the growth of the maxima of its successive derivatives in I. The following simple theorem characterizes the class of i. d. functions composed of all analytic functions in a given interval.

THEOREM I. *A necessary and sufficient condition, in order that an i. d. function, $f(x)$, defined in $I\equiv[a, b]$, be analytic in this interval, is that there exist a positive constant k, depending only on f, $(k=k(f))$, such that*

(1) $$|f^{(n)}(x)|<k^n n!\,(n\geqq 1,\ x\epsilon[a, b]).$$

The condition is necessary. Let, indeed, $f(x)$ be analytic in I. Then to every $x_0 \epsilon I$ there corresponds a Taylor series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

which converges to $f(x)$ in a certain interval around x_0. Consider now the series in the complex variable $z=x+iy$:

$$\sum a_n(z-x_0)^n. \tag{2}$$

This series defines a function holomorphic in a certain open circle with the center x_0, which we shall denote by C_{x_0}. The function defined by (2) is equal, for real values of x, to $f(x)$. By the Borel-Lebesgue theorem it is possible to choose a finite number of points x_0, belonging to I, such that every point of I is interior to one of the corresponding circles C_{x_0}, two consecutive such circles having, thus, a common part. Let D be the domain formed by these circles, and let us denote by C the frontier of D. Since two expansions (2) corresponding to two circles with a common part are equal at the real segment of their common part, we see that the expansions (2) define a holomorphic function $f(z)$ in D, equal to $f(x)$ on I.

Denote then by r the smallest distance from C to $[a, b]$. Obviously $r>0$. About each point $x \epsilon I$ consider a closed circle C_x' with radius $\frac{r}{2}$. The set of circles C_x' forms a closed connected region D'. The function $f(z)$ is holomorphic and bounded in D': $|f(z)|<M (z \epsilon D')$. We have then, by the Cauchy integral formula:

$$f^{(n)}(x)=\frac{n!}{2\pi i}\oint_{C_x'}\frac{f(z)dz}{(z-x)^{n+1}},$$

and therefore

$$|f^{(n)}(x)|<M\left(\frac{2}{r}\right)^n n!.$$

Thus we have established for $n \geqq 1$, the inequality (1), i. e.,

$$|f^{(n)}(x)| < k^n n! \quad (x \epsilon I).$$

The condition is sufficient. Suppose that (1) is satisfied. For every x_0 and x, belonging both to I, we may write

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)(x-x_0)^{n-1}}{(n-1)!}$$

$$+ \frac{f^{(n)}\big(x_0 + \theta(x-x_0)\big)(x-x_0)^n}{n!} \quad (0 < \theta < 1).$$

But since $x_0 + \theta(x - x_0) \epsilon I$, we have

$$\left|f^{(n)}\big(x_0 + \theta(x-x_0)\big)\right| < k^n n!$$

and therefore

$$\left| f(x) - f(x_0) - f'(x_0)(x-x_0) \cdots - \frac{f^{(n-1)}(x_0)(x-x_0)^{n-1}}{(n-1)!} \right|$$

$$< \frac{k^n n! |x - x_0|^n}{n!} = (k|x-x_0|)^n.$$

We see then that, if

$$k|x - x_0| \leqq \rho < 1, \ x_0 \epsilon I, \ x \epsilon I,$$

the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

converges to $f(x)$, and this function is therefore analytic in I.

2. OTHER CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

In studying the heat equation

(3) $$\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0,$$

Gevray (8)[1] introduces, in a natural way, i. d. functions satisfying, in an interval, the inequalities

(4) $$|f^{(n)}(x)| < k^n (2n)! \quad (n \geqq 1).$$

[1]For bibliographical references see p. 142.

By the Cauchy-Kowalewski theorem, there exists one, and only one, function $u(x, y)$, analytic in both variables x and y, satisfying (3), and such that

$$u(x, 0) = u_0(x),$$

$$\frac{\partial u}{\partial y}(x, 0) = u_1(x), \tag{5}$$

the functions $u_0(x)$, $u_1(x)$ being any two given analytic functions of x.

Thus the following question arises: suppose $u_0(x) \equiv 0$, what are the general conditions which $u_1(x)$ must satisfy, in order that there exist a function $u(x, y)$ satisfying (3) and the two conditions (5)?

We shall suppose that $u(x, y) = -u(x, -y)$. S. Bernstein (8) has proved that $u(x, y)$ is analytic in y. Therefore, for every x, in the neighborhood of $y = 0$, the following expansion is satisfied:

$$u(x, y) = u_1(x)y + \frac{u_3(x)}{3!}y^3 + \cdots + \frac{u_{2p+1}(x)}{(2p+1)!}y^{2p+1} + \cdots .$$

From (3) it follows, by differentiation, and by mathematical induction, that, for every (x, y),

$$\frac{\partial^{2p+1}u}{\partial y^{2p+1}} = \frac{\partial^{p+1}u}{\partial x^p \partial y} = \frac{\partial^p}{\partial x^p}\left(\frac{\partial u}{\partial y}\right)^1.$$

And, particularly, for $y = 0$, we have, since $\frac{\partial u}{\partial y}(x, 0) = u_1(x)$,

$$\frac{\partial^{2p+1}u}{\partial y^{2p+1}}(x, 0) = \frac{d^p u_1(x)}{dx^p}. \tag{6}$$

Using then, in an obvious manner, the Cauchy integral formula, we see immediately that, if x is in a closed interval $[a, b]$,

$$\left|\frac{\partial^{2p+1}u}{\partial y^{2p+1}}(x,0)\right| \leqq \rho^{2p+1}(2p+1)!,$$

where ρ is a positive constant. For it is obvious that when y is in a certain circle about the origin, and $x \epsilon [a, b]$, then

[1]The partial derivatives satisfying conditions of continuity.

$|u(x, y)|<M$, and it is sufficient to express the value of $\frac{\partial^{2p+1}u}{\partial y^{2p+1}}(x, 0)$, by Cauchy's formula, in order to have the desired estimate.

Thus, by (6), the function $u_1(x)$ must satisfy the inequalities

$$|u_1^{(p)}(x)| \leqq \rho^{2p+1}(2p+1)!<k^p(2p)! \quad (p\geqq 1),$$

where k is a positive constant.

The class of all functions satisfying (4) contains, by Theorem I, all analytic functions, in $[a, b]$. But does this new class contain other i. d. functions than analytic ones? The answer is affirmative, since the function $f(x)$, formed on page 2, which, as we have seen, is not analytic on any interval, belongs to this class.

It is clear, indeed, that

$$|f^{(n)}(x)| \leqq \sum_{m=1}^{\infty} e^{-m^{\frac{1}{4}}}m^n < \underset{m\geqq 1}{\mathrm{Max}}(e^{-m^{\frac{1}{4}}}m^{n+2}) \sum_{m=1}^{\infty}\frac{1}{m^2}$$
$$< C \underset{x>0}{\mathrm{Max}}\,(e^{-x^{\frac{1}{4}}}x^{n+2}) = ce^{-2(n+2)}(2(n+2))^{2(n+2)}<k^n(2n)!.$$

Considering equations of the type

$$\frac{\partial^r u}{\partial y^r}-\frac{\partial^s u}{\partial x^s}, \quad (r>s),$$

Gevray (8) introduced i. d. functions satisfying, in any interval, the inequalities

$$(7) \qquad |f^{(n)}(x)|<k^n\Gamma\left(\frac{r}{s}n\right), \quad (n\geqq 1).$$

Such a class, whatever may be the integers r and s (if only $r>s$), contains all analytic functions, but also functions which are not analytic. This statement, likewise the analogous one regarding the class defined by (4), follows from a general theorem, which will be proved later, although it is easy, as for the class (4), to give a simple proof, also in this particular case.

3. GENERAL CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

The class of analytic functions, in a given closed interval I, and the classes (4) and (7), introduced by considerations of theoretical physics, and containing functions, which are not analytic, are particular cases of a family of classes defined in the following manner:

Consider a positive sequence of numbers $\{M_n\}$, $(M_1, M_2, \cdots)$, and a given closed interval $I \equiv [a, b]$. The class of all i. d. functions defined in I and such that to each of them corresponds a constant $k = k(f)$, such that

$$|f^{(n)}(x)| < k^n M_n, \quad (n \geqq 1, x \epsilon I),$$

will be denoted by $C_{\{M_n\}}$.

It is obvious that, if two sequences $\{M_n\}$ and $\{M_n'\}$ are such that there exists a positive constant α with

$$M_n < \alpha^n M_n' \quad (n \geqq 1),$$

then the class $C_{\{M_n\}}$ is a *subclass* of the class $C_{\{M_n'\}}$, i. e., every function of $C_{\{M_n\}}$ belongs also to $C_{\{M_n'\}}$; since, if $f \epsilon C_{\{M_n\}}$ then

$$|f^{(n)}(x)| < k^n M_n < (\alpha k)^n M_n' \quad (n \geqq 1),$$

and $f \epsilon C_{\{M_n'\}}$.

Therefore, if two positive constants α and β exist such that

$$\beta^n M_n' < M_n < \alpha^n M_n', \tag{8}$$

then the two classes $C_{\{M_n\}}$ and $C_{\{M_n'\}}$ are equivalent, that is to say, every function of either of these classes belongs also to the other.

For instance, the classes $C_{\{n!\}}$ and $C_{\{n^n\}}$ are equivalent, and are composed of all analytic functions in any given interval and only of such functions.

The classes defined by (4) and (7) may be denoted respectively by $C_{\{(2n)!\}}$ and $C_{\left\{\Gamma\left(\frac{r}{s}n+1\right)\right\}}$, the last one being equivalent to $C_{\left\{\Gamma\left(\frac{r}{s}n\right)\right\}}$.

In the first part of these lectures we shall be concerned mostly with the general structure of classes $C_{\{M_n\}}$, where $\{M_n\}$ is any sequence of positive numbers.

4. THE PROBLEM OF EQUIVALENCE OF TWO CLASSES

The condition (8), which may be written in the following manner:

$$0 < \varliminf_{n=\infty} \left(\frac{M_n'}{M_n}\right)^{1/n} \leqq \varlimsup_{n=\infty} \left(\frac{M_n'}{M_n}\right)^{1/n} < \infty, \tag{9}$$

is sufficient for the equivalence of two classes, $C_{\{M_n\}}$ and $C_{\{M_n'\}}$. But this elementary condition is surely not necessary for equivalence.

Consider, for instance, the class $C_{\{M_n\}}$, where the sequence $\{M_n\}$ is defined in the following manner:

$$\begin{aligned} &M_n = n! \quad \textit{for } n \textit{ even}, \\ &M_n = \textit{any fixed number} > n!, \textit{for } n \textit{ odd}. \end{aligned} \tag{10}$$

This class is equivalent to the class $C_{\{n!\}}$. Indeed, let $f(x)$ belong in $I \equiv [a, b]$ to $C_{\{M_n\}}$, then

$$|f^{(n)}(x)| < k^n M_n \quad (n \geqq 1).$$

For any two points x_0 and x_1 of $[a, b]$, we can write

$$f^{(n)}(x_1) = f^{(n)}(x_0) + f^{(n+1)}(x_0)(x_1 - x_0) + \frac{f^{(n+2)}(x')(x_1 - x_0)^2}{2},$$

where x' is a point situated between x_0 and x_1. Let us then take x_0 arbitrarily and choose x_1 in such a manner that

$$|x_1 - x_0| \geqq \frac{b-a}{2} = \gamma.$$

Then

$$\begin{aligned} |f^{(n+1)}(x_0)| &= \left| \frac{f^{(n)}(x_1) - f^{(n)}(x_0)}{x_1 - x_0} - f^{(n+2)}(x') \frac{x_1 - x_0}{2} \right| \\ &\leqq \frac{2M_n}{\gamma} + \gamma M_{n+2}. \end{aligned}$$

Hence, if n is even,

$$|f^{(n+1)}(x_0)| < \frac{2k^n n!}{\gamma} + \gamma k^{n+2}(n+2)! < k_1^{n+1}(n+1)!,$$

where k_1 is a constant. If now $k_2 = max\ (k, k_1)$, we see that

$$|f^{(n)}(x)| < k_2^n n! \quad (n \geqq 1).$$

Therefore, every function of $C_{\{M_n\}}$ belongs to $C_{\{n!\}}$. The converse is obviously true, and the two classes $C_{\{M_n\}}$ and $C_{\{n!\}}$ are equivalent. And yet if, for instance, for n odd,

$$M_n = (n!)^n,$$

then

$$\overline{\lim_{u=\infty}} \left(\frac{M_n}{n!}\right)^{1/n} = \infty.$$

In a general manner, it is possible to construct in many different ways a class $C_{\{M_n'\}}$ equivalent to a given class $C_{\{M_n\}}$ and such that there is no simple relationship between the behaviour of the sequence $\{M_n\}$ and that of the sequence $\{M_n'\}$. Thus arises the problem which may be stated as follows:

To give necessary and sufficient conditions characterizing the relationship between the two sequences $\{M_n\}$ and $\{M_n'\}$ in order that the two classes $C_{\{M_n\}}$ and $C_{\{M'_n\}}$ be equivalent.

It is obvious of course that this problem will be solved if we solve the following one:

To give necessary and sufficient conditions, bearing on the sequences $\{M_n\}$ and $\{M_n'\}$, in order that the class $C_{\{M_n\}}$ be a subclass of the class $C_{\{M'_n\}}$. That is to say, in order that every function belonging to $C_{\{M_n\}}$ belong also to $C_{\{M'_n\}}$.

For, if such conditions are found, we have to add to these conditions the conditions obtained by exchanging the rôles of the two sequences $\{M_n\}$, $\{M_n'\}$, in order to have conditions for equivalence of the classes.

In the special, but important, case when $M_n' = n!$ this problem can be translated in the following manner:

To give necessary and sufficient conditions, bearing on the sequence $\{M_n\}$, *in order that every function* $f(x)$ *of* $C_{\{M_n\}}$ *be analytic.*

A class $C_{\{M_n\}}$ containing only analytic functions will be called an *analytic class*. Therefore, in the preceding problem, we are looking for conditions on $\{M_n\}$ in order that $C_{\{M_n\}}$ be an analytic class.

This last problem, and the general problem of equivalence, were proposed by Carleman (**2**) in 1926, but they have found their solution only in the last few years. This solution depends largely on a new principle, which we shall call the "*principle of regularization of sequences*," introduced by the author of these lectures (**9**) and which will be treated in the next paragraph.

5. PRINCIPLE OF REGULARIZATION OF SEQUENCES

In the example of the class $C_{\{M_n\}}$, where the sequence $\{M_n\}$ satisfies (10), the sequence of indices was divided into two subsequences: $\{n_i\}$ and $\{m_j\}$ in such a manner that when the values of the quantities M_{n_i} were known, the class $C_{\{M_n\}}$ was perfectly determined, no matter what values the M_{m_j} might have had, provided only that they were greater than certain quantities depending only upon the sequence $\{M_{n_i}\}$. In the particular example we have been concerned with, $\{n_i\}$ was the sequence of even integers and $\{m_j\}$ that of odd integers. But this particular choice of $\{n_i\}$ was due to the regularity of the set of values taken by the M_n, with even indices. Such a splitting of the sequence $\{M_n\}$ into two categories $\{M_{n_i}\}$ and $\{M_{m_j}\}$ is possible, in the most general case, but, in the general case, the values of the indices n_i of the M_{n_i}, upon which the M_{m_j} depend, and the smallest values, by which the M_{m_j} may be replaced, are determined by a geometrical process, for which we shall give an analytic interpretation, which process constitutes the principle of regularization in question (9).

We shall denote by $\omega(t)$ a positive function of t, defined for $t \geqq 0$, such that $\omega(0) \geqq 1$, and having one of the three properties:

(1) $\omega(t)$ is a continuous function of t, strictly increasing to $+\infty$ when t tends to $+\infty$.

(2) there is a value $t=t_0>0$, such that for $t<t_0$ $\omega(t)$ is continuous and strictly increasing to $+\infty$ when t tends to t_0 increasingly. For $t \geqq t_0 : \omega(t) = \infty$.

(3) $\omega(t)$ is identically equal to $+\infty$.

Consider, in the xy plane, a sequence of points $\{P_n\}$, the coordinates of the point P_n being $x=n$, $y=\alpha_n$ $(n \geqq 1)$. The quantity α_n may take the value $+\infty$ (but not $-\infty$): we shall say, then, that the corresponding point P_n lies, on the line $x=n$, at infinity. At any rate, we shall suppose that there exists an infinity of points P_n which have a finite ordinate. We shall also suppose, for convenience in exposition, that α_1 is finite.

We shall "*regularize*" the sequence of points $\{P_n\}$ or, what amounts to the same thing, the sequence of numbers $\{\alpha_n\}$, *with respect* to the function $\omega(t)$.

Regularization with respect to a function $\omega(t)$ satisfying (2) or (3) will be useful only if the sequence $\{\alpha_n\}$ is such that

$$\lim \frac{\alpha_n}{n} = \infty.$$

Since this simplifies the details in the exposition, we shall suppose that only sequences of this type will be regularized with respect to a function $\omega(t)$ satisfying (2) or (3).

Denote by Σ_t the closed strip defined, in the xy plane, by the inequality: $0 \leqq x \leqq \omega(t)$ (the half plane $x \geqq 0$, if $\omega(t) = \infty$). Consider the straight line Δ_t with the slope t such that there is no point P_n in Σ_t below Δ_t, but there exists at least one point P_n on the part of Δ_t which is in Σ_t (i. e. there is no point P_n with $1 \leqq n \leqq \omega(t)$ below Δ_t, but there exists such a point on Δ_t). The part of Δ_t which is in Σ_t will be denoted by D_t. The projection of D_t on Ox is the closed interval $0 \leqq x \leqq \omega(t)$ (the ray $x \geqq 0$, if $\omega(t) = \infty$).

The existence of D_t, for every $t \geqq 0$, is assured, in the case when $\omega(t) < \infty$, by the circumstance that α_1 is finite and $\alpha_n > -\infty$ $(n \geqq 1)$, and, in the case when $\omega(t) = \infty$, also by the supplementary condition that $\lim \dfrac{\alpha_n}{n} = \infty$, or what amounts to the same thing, that for every given t we have $\alpha_n > nt$ for large values of n. Denote by $\{P_{n_i}\}$ the sequence of points belonging to $\{P_n\}$, each of which is situated on at least one D_t. The sequence $\{n_i\}$ (the set of abscissas of points P_n which are on at least one D_t) is infinite, for it is immediately seen that, if n_{i_0} were the greatest index n such that P_n is on a D_t, all the points P_n with $n > n_{i_0}$ would have an ordinate $\alpha_n = \infty$, contrary to our hypotheses. Every point P_{n_i} $(i = 1, 2, \cdots)$ is thus on a certain D_t. No other point P_n is on such a segment.

The sequence $\{n_i\}$ will be called the *principal sequence of indices of the sequence* $\{P_n\}$, *with respect to* $\omega(t)$. The sequence $\{P_{n_i}\}$ will be called the *principal subsequence of the sequence* $\{P_n\}$ *with respect to* $\omega(t)$. In what precedes, we may often replace the sequences of points $\{P_{n_i}\}$ or $\{P_n\}$ by the corresponding sequences of their ordinates: $\{\alpha_{n_i}\}$ or $\{\alpha_n\}$.

A point P_{n_i}, of the principal subsequence lies, *a priori*, on many D_t. Denote by T_i the set of all such values t. Obviously, if t_1 and t_2, with $t_1 < t_2$, belong to T_i, i. e., if P_{n_i} lies on D_{t_1} and D_{t_2}, then every t such that $t_1 < t < t_2$ belongs to T_i. Therefore T_i forms an interval. It is also obvious that the lower extremity of this interval belongs to T_i, but not necessarily the upper extremity. Denote the last quantity by τ_i, that is to say,

$$\tau_i = \overline{\mathrm{Bd}}\ T_i.$$

It is seen immediately that τ_i *increases with* i.

We have said that τ_i does not necessarily belong to T_i, that is, T_i is not necessarily closed at the right. This means that there may exist a point P_n which is in Σ_{τ_i} below the straight line passing through P_{n_i} and with the slope τ_i. In other words the part of this straight line which is in Σ_{τ_i} (i. e., for which $0 \leqq \alpha \leqq \omega(\tau_i)$) is not the segment D_{τ_i}. This can only occur if $\omega(\tau_i) < \infty$; thus, since $\omega(t)$ is then continuous for $t \leqq \tau_i$, if there exists a point P_n in Σ_{τ_i} below the straight line with the slope τ_i passing through P_{n_i}, this point is necessarily such that $n = \omega(\tau_i)$. It is also obvious that, for values $t > \tau_i$ near enough to τ_i, D_t passes through this point P_n. That is to say, $P_n = P_{n_{i+1}}$. Hence in this case

$$n_{i+1} = \omega(\tau_i).$$

No point, other than the point $P_{n_{i+1}}$, can be in Σ_{τ_i} below the straight line with the slope τ_i passing through P_{n_i}.

Denote by L_i the part of the straight line with the slope τ_i passing through P_{n_i}, of which the projection on Ox is the segment (closed on the left-hand and open on the right-hand)

$$n_i \leqq x < n_{i+1}.$$

L_0 will be the part of the segment D_0 for which $1 \leqq x \leqq n_1$.

We shall call the set of all the segments L_i ($i = 0, 1, 2, \cdots$) the *ω-base of the sequence* $\{P_n\}$ (or of the sequence $\{\alpha_n\}$). The end-point of a L_i with the greatest abscissa belongs to the base only, if it is also the beginning-point (that is to say the point with the smallest abscissa) of L_{i+1}.

Therefore, it is clear that the base is composed of a succession of polygonal lines, disjointed from each other, the consecutive sides of each polygonal line having increasing slopes (going from left to right).

If Π_j and Π_{j+1} are two consecutive polygonal lines composing the base, the right end-point of Π_j (with the greatest abscissa) does not belong to the base, and Π_{j+1} begins at a point which has the same abscissa as the end-point of Π_j, but an ordinate *smaller* than that of the end-point of Π_j.

The common abscissa of the end-point of Π_j and the beginning point of Π_{j+1} is a principal index: say n_k. The slope τ_k of the segment L_k, with which Π_{j+1} begins, is greater than the slope τ_{k-1} of the last side L_{k-1} of Π_j. A principal index, which is the common abscissa of two polygonal lines (in the case of the two polygonal lines Π_j and Π_{j+1}, this principal index is n_k), will be called *an index of discontinuity*.

It may happen, of course, that there is no index of discontinuity. Then the ω-base is a convex, continuous, polygonal line with sides having increasing slopes.

Denote by $\alpha_n^{(\omega)}$ the ordinate of the point of the ω-base of $\{P_n\}$ which has the abscissa n.

The sequence $\{\alpha_n^{(\omega)}\}$ will be called the *regularized sequence of the sequence* $\{\alpha_n\}$ *with respect to* $\omega(t)$. And, denoting by $P_n^{(\omega)}$ the point with abscissa n and ordinate $\alpha_n^{(\omega)}$, we shall call the sequence $\{P_n^{(\omega)}\}$ *the regularized sequence of the sequence* $\{P_n\}$ *with respect to* $\omega(t)$.

By definition, the principal points P_{n_i} of the sequence $\{P_n\}$ remain invariant by the regularization, i. e., $P_{n_i}^{(\omega)}=P_{n_i}$. It is also obvious that for every $n \geqq 1$ we have $\alpha_n^{(\omega)} \leqq \alpha_n$, and that the equality holds only if $n=n_i$ $(i=1, 2, \cdots)$. If we consider, together with the sequence $\{\alpha_n\}$ the sequence $\{\alpha_n'\}$, and if, for the principal sequence of indices $\{n_i\}$ of the sequence $\{\alpha_n\}$, $\alpha_{n_i}'=\alpha_{n_i}$, and if $\alpha_n' \geqq \alpha_n$ $(n=1, 2, \cdots)$, then $\alpha_n'^{(\omega)}=\alpha_n^{(\omega)}$ $(n \geqq 1)$.

It is easy to see that, if $\omega(t)$ and $\omega_1(t)$ are two functions both satisfying one of the three conditions required above, and such that for $t \geqq 0$ we have $\omega_1(t) \leqq \omega(t)$, then, for every $n \geqq 1$, we have $\alpha_n^{(\omega)} \leqq \alpha_n^{(\omega_1)}$.

This shows that if $\omega_1(t) < \omega(t)$, $(t>0)$, then for every given sequence $\{P_n\}$ the ω_1-base is nearer to the points of this sequence than the ω-base. This intuitive manner of expressing the fact which precedes may also be interpreted in the following no less intuitive manner: the smaller the function $\omega(t)$, for every $t>0$, the more the regularized sequence resembles the sequence itself. But it should be remarked that, if we decrease $\omega(t)$, for every $t>0$, we increase the number of indices of discontinuity of the base. In that case we may say that the regularity of the regularized sequence is decreased. Thus, a gain in approaching more intimately a given sequence results in a loss of regularity in the regularized sequence.

It is clear that, for different purposes, the admixture of these two factors, regularity and intimacy of connection with the primitive sequence, has to be properly chosen. That is to say, for different purposes, $\omega(t)$ has to be differently chosen.

Let us now give an analytic interpretation of regularization.

$\{\alpha_n\}$ being a sequence having the properties described above ($\alpha_1 < \infty$, $\alpha_n > -\infty$ for every n, $\alpha_n < \infty$ for an infinity of n, and, if $\omega(t) = \infty$ for $t > t_0$, $\lim \frac{\alpha_n}{n} = \infty$), the function, defined for $t \geqq 0$ by the equality:

$$A(t) = \operatorname*{Max}_{n \leqq \omega(t)} (nt - \alpha_n),^1$$

will be called the *ω-trace function of the sequence* $\{\alpha_n\}$. The

[1]The *maximum* is evidently taken with respect to n. For every t, n takes all the positive integral values which do not exceed $\omega(t)$.

reason for this denomination will appear later. This function $A(t)$ exists, since, if $\omega(t) < \infty$, $A(t)$ is the greatest value of a finite set of numbers; and if $\omega(t) = \infty$, the existence of $A(t)$ is assured by the condition $\lim \frac{\alpha_n}{n} = \infty$ (which is supposed to hold when $\omega(t) = \infty$ for $t \geqq t_0$), since then $nt - \alpha_n$ tends to $-\infty$ when n tends to ∞.

Since $\omega(t)$ increases with t, it is plain that $A(t)$ increases with t.

The trace function $A(t)$ is such that $A(t+0)$ exists and $A(t+0) = A(t)$ for every $t \geqq 0$. The existence of $A(t+0)$ is obvious since $A(t)$ is an increasing function. Denote then by n_ϵ an integer such that $A(t+\epsilon) = n_\epsilon(t+\epsilon) - \alpha_{n_\epsilon}$ ($\epsilon > 0$), $n_\epsilon \leqq \omega(t+\epsilon)$. If $t \geqq 0$ is given, the set of the integers n_ϵ, each satisfying the preceding equality for all ϵ satisfying the inequality $0 < \epsilon < k$, is finite, since if $\omega(t+k) < \infty$, $n_\epsilon \leqq \omega(t+k)$, and if $\omega(t+k) = \infty$, $n(t+k) - \alpha_n$ tends to $-\infty$ when $n \to \infty$. Therefore there exists a sequence $\epsilon_i \downarrow 0$[1] for which n_{ϵ_i} takes the same value, say m. But then, since $m \leqq \omega(t+\epsilon_i)$ ($i = 1, 2, \cdots$), we have also $m \leqq \omega(t)$, and $A(t+\epsilon_i) = m(t+\epsilon_i) - \alpha_m \downarrow mt - \alpha_m$. Thus $A(t+0) = mt - \alpha_m \leqq A(t)$. And, since $A(t)$ is an increasing function, $A(t+0) = A(t)$. It is also obvious that $A(t-0)$ exists. Generally $A(t-0)$ need not be equal to $A(t)$, but if $\omega(t) = \infty$, then $A(t-0) = A(t)$, and $A(t)$ is continuous at t. Indeed, if $A(t) = mt - \alpha_m$, for $\epsilon > 0$ and sufficiently small, $\omega(t-\epsilon) > m$, and, therefore, $A(t-\epsilon) \geqq m(t-\epsilon) - \alpha_m \to mt - \alpha_m = A(t)$, i. e., $A(t-0) \geqq A(t)$, and, since $A(t)$ is an increasing function, $A(t-0) = A(t)$.

Denote by $m(t)$ the greatest integer $n \leqq \omega(t)$ such that $A(t) = nt - \alpha_n$. $m(t)$ is an increasing function. Indeed we have by definition if $t_1 < t_2$: $m(t_1)t_2 - \alpha_{m(t_1)} \leqq m(t_2)t_2 - \alpha_{m(t_2)}$

[1] $a_i \downarrow b$ means that a_i tends decreasingly to b, when i tends to ∞; $a_i \uparrow b$ means that a_i tends increasingly to b, when i tends to ∞.

$=A(t_2)$, that is to say, $[m(t_1)-m(t_2)]t_2 \leqq \alpha_{m(t_1)}-\alpha_{m(t_2)}$. If we had $m(t_1)>m(t_2)$, we should also have $m(t_1)t_1-\alpha_{m(t_1)} \geqq m(t_2)t_1 -\alpha_{m(t_2)}$, i. e., $\alpha_{m(t_1)}-\alpha_{m(t_2)} \leqq [m(t_1)-m(t_2)]t_1$, and therefore together with the preceding inequality $[m(t_1)-m(t_2)]t_2 \leqq [m(t_1)-m(t_2)]t_1$, which proves, since $t_2>t_1$, that $m(t_1) -m(t_2)<0$, contrary to the supposition. $m(t+0)$ thus exists. And this quantity is equal to $m(t)$ since $A(t+0)=m(t+0)t -\alpha_{m(t+0)}=A(t)=m(t)t-\alpha_{m(t)}$, and if $m(t+0)$ were greater than $m(t)$, $A(t)$ could be written as equal to $nt-\alpha_n$ with $n=m(t+0)>m(t)$, $n=m(t+0) \leqq \omega(t+0)=\omega(t)$, contrary to the definition of $m(t)$.

The function $m(t)$ tends to ∞ : $m(t) \uparrow \infty$, with t. Indeed if $t \geqq \text{Max}\,(\alpha_2-\alpha_1, \alpha_3-\alpha_2, \cdots, \alpha_p-\alpha_{p-1})$, then $m(t) \geqq \min(p, [\omega(t)])$,[1] since then $pt-\alpha_p \geqq (p-1)t-\alpha_{p-1} \geqq (p-2)t-\alpha_{p-2} \cdots, \geqq t-\alpha_1$, and $m(t)$ cannot be smaller than the greatest of the integers $1, 2, \cdots, p$, which are smaller than or equal to $\omega(t)$. Thus $m(t)$ tends to infinity.

All the properties of $m(t)$ show that there exists a sequence of numbers $t_0=0, t_1, t_2, \cdots, t_k, \cdots$, tending increasingly to infinity such that in every interval $t_k \leqq t < t_{k+1}$, $m(t)=m(t_k)$, $m(t_k)$ being a positive integer tending increasingly to infinity. Therefore, if $t_k \leqq t < t_{k+1}$,

$$A(t)=m(t_k)t-\alpha_{m(t_k)},$$

and

$$A(t)-A(t_k)=m(t_k)(t-t_k).$$

Thus, in this interval

$$\frac{d^+A(t)}{dt}=m(t_k).^2 \tag{11}$$

We have said that generally $A(t-0) \neq A(t)$. This can happen evidently only at points $t=t_k$. If at such a point

[1] $[a]$ denotes the greatest integer smaller than or equal to a.

[2] $\frac{d^+f(t)}{dt}$ denotes the right-hand derivative of $f(t)$ at the point t.

$A(t)$ is discontinuous, we have necessarily $A(t_k)-A(t_k-0)>0$. Let us denote by $d(t_k)$ the quantity $A(t_k)-A(t_k-0)$, and consider the function

$$a(t)=\sum_{t_k\leqq t} d(t_k).$$

The function $a(t)$ may also be written as a Stieltjes integral

$$a(t)=\int_0^t dV(\tau), \tag{12}$$

$V(\tau)$ being an increasing function, constant in every interval $t_k\leqq t<t_{k+1}$, with $V(t_k)-V(t_k-0)=d(t_k)$.

From (11) and (12) it follows immediately that if $t>t'$

$$A(t)=A(t')+\int_{t'}^t m(\tau)d\tau+\int_{t'}^t dV(\tau). \tag{13}$$

We recall that, if $\omega(t^*)=\infty$, $A(t^*-0)=A(t^*)$, and therefore for every $t_k\geqq t^*$, $d(t_k)=0$. $V(t)$ is therefore constant for $t\geqq t^*$. In the particular case, when $\omega(t)$ is identically equal to $+\infty$, we have

$$A(t)=A(t')+\int_{t'}^t m(\tau)d\tau, \tag{14}$$

for every pair of values t' and t, $t>t'$.

It is also important to make the following remarks:

If one changes a finite number of terms in the sequence $\{\alpha_n\}$, the ω-trace function, $A(t)$, remains unchanged for large values of t. That is to say, we may suppose that the first m terms, $\alpha_1, \alpha_2, \cdots, \alpha_m$, are all equal to infinity, and $A(t)$ is, for large values of t, the same as for the primitive sequence:

$$A(t)=\operatorname*{Max}_{m\leqq n\leqq\omega(t)}(nt-\alpha_n), \quad (t>t_0).$$

This follows immediately from the remark that $m(t)\uparrow\infty$.

The sequence $\{\alpha_n\}$ will be called the *ω-generatrix of the ω-trace function, $A(t)$.* An ω-trace function may admit an infinity of ω-generatrices.

We shall now prove the following Lemma.

LEMMA I. *Let $A(t)$ be an ω-trace function. There exists a sequence $\{a_n\}$, which is the smallest ω-generatrix of $A(t)$. That is to say that, if $\{\alpha_n\}$ is any ω-generatrix of $A(t)$, we have, for every $n \geqq 1$,*

$$a_n \leqq \alpha_n. \tag{15}$$

This smallest ω-generatrix of $A(t)$ is given by the relationship

$$a_n = \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - A(t)).\text{[1]} \tag{16}$$

$\{\alpha_n\}$ being any ω-generatrix of $A(t)$, that is to say,

$$A(t) = \operatorname*{Max}_{n \leqq \omega(t)} (nt - \alpha_n), \tag{17}$$

we have to prove that $\{a_n\}$ defined by (16) is also a ω-generatrix of $A(t)$, and, then, that (15) holds. We have then to prove first of all that

$$A(t) = \operatorname*{Max}_{n \leqq \omega(t)} \{nt - \overline{\mathrm{Bd}}_{\omega(s) \geqq n} (ns - A(s))\}. \tag{18}$$

We have for $t \geqq 0$,

$$\operatorname*{Max}_{n \leqq \omega(t)} \{nt - \overline{\mathrm{Bd}}_{\omega(s) \geqq n} (ns - A(s))\}$$
$$\leqq \operatorname*{Max}_{n \leqq \omega(t)} \{nt - (nt - A(t))\} = A(t).$$

We have also by (17)

$$\operatorname*{Max}_{n \leqq \omega(t)} \{nt - \overline{\mathrm{Bd}}_{\omega(s) \geqq n} (ns - A(s))\}$$
$$= \operatorname*{Max}_{n \leqq \omega(t)} \{nt - \overline{\mathrm{Bd}}_{\omega(s) \geqq n} (ns - \operatorname*{Max}_{m \leqq \omega(s)} (ms - \alpha_m))\}$$
$$\geqq \operatorname*{Max}_{n \leqq \omega(t)} \{nt - \overline{\mathrm{Bd}}_{\omega(s) \geqq n} (ns - (ns - \alpha_n))$$
$$= \operatorname*{Max}_{n \leqq \omega(t)} (nt - \alpha_n) = A(t).$$

Thus (18) is proved. It results now from (16) that

$$a_n = \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - \operatorname*{Max}_{m \leqq \omega(t)} (mt - \alpha_m))$$
$$\leqq \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - (nt - \alpha_n)) = \alpha_n.$$

[1]The least upper bound is taken with respect to t, when t takes all values such that $\omega(t) \geqq n$.

The Lemma is therefore proved.

The following lemma characterizes, in a new manner, the regularized sequence of a sequence $\{\alpha_n\}$ with respect to $\omega(t)$.

LEMMA II. *The regularized sequence* $\{\alpha_n^{(\omega)}\}$ *of the sequence* $\{\alpha_n\}$ *is the smallest* ω*-generatrix of the* ω*-trace function of the sequence* $\{\alpha_n\}$.

Consider as above, in the xy plane, the points $P_n \equiv (n, \alpha_n)$. The ω-trace function of $\{\alpha_n\}$ is given by

$$A(t) = \operatorname*{Max}_{n \leqq \omega(t)} (nt - \alpha_n),$$

where the quantity $\alpha_n - nt$ represents the y-intercept of the straight line having the slope t and passing through P_n. Thus the quantity $-A(t)$ represents the y-intercept of the segment D_t (see page 13). This is why we call $A(t)$ the ω-trace function.

The point $P_n^{(\omega)} \equiv (n, \alpha_n^{(\omega)})$ is on the segment L_i which begins at the point P_{n_i}, n_i being the greatest principal index smaller than or equal to n,[1] and the slope of L_i being equal to $\tau_i = \overline{\mathrm{Bd}}\ T_i$, where T_i is the set of the values of t such that P_{n_i}, is on D_t. On the other hand, if $\omega(t) \geqq n$, the point $(n, nt - A(t))$ is on the line D_t. Recalling then the form of the ω-base of regularization, one sees immediately that the least upper bound of the ordinates of intersection of all the lines $D_t(\omega(t) \geqq n)$ with the line $x = n$ is precisely the ordinate of intersection of L_i with the line $x = n$. But this least upper bound is

$$a_n = \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - A(t)).$$

Thus $a_n = \alpha_n^{(\omega)}$ and the lemma is proved. It is clear that $(\alpha_n^{(\omega)})^{(\omega)} = \alpha_n^{(\omega)}$. In other words: the ω-regularized sequence of the ω-regularized sequence of the sequence $\{\alpha_n\}$ is the ω-regularized sequence of $\{\alpha_n\}$. It is clear that, if for every

[1]If $n < n_1$, $P_n^{(\omega)}$ is on L_0.

$n \geqq 1$ we have $\alpha_n - \beta_n = C$ (constant), then it follows from the proved Lemma that $\alpha_n^{(\omega)} - \beta_n^{(\omega)} = C$.

Let us also remark that, if we change a finite number of terms in the sequence $\{\alpha_n\}$, only a finite number of terms change in $\{\alpha_n^{(\omega)}\}$. For the case (when $\omega(t) < \infty$ for $t \geqq 0$) this results from the formula

(19) $$\alpha_n^{(\omega)} = a_n = \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - A(t)),$$

since, from $\omega(t) \geqq n$, it follows that if n is sufficiently large t is large, and when t is large we have seen on page 19 that $A(t)$ remains unchanged after we have changed a finite number of α_n. If now $\omega(t) = \infty$ for t sufficiently large, our assertion follows from $\lim \frac{\alpha_n}{n} = \infty$, and from the geometrical construction of the points $\{P_n^{(\omega)}\}$.

We shall now prove the following Lemma which will often be useful.

Lemma III. *If $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences such that there exists a positive constant γ satisfying the condition*

(20) $$\alpha_n^{(\omega)} \leqq \beta_n + \gamma n \quad (n \geqq 1),$$

then, denoting by $A(t)$ and $B(t)$ the respective ω-traces of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we have for $t \geqq \gamma$,

(21) $$A(t) \geqq B(t - \gamma).$$

If there exists a positive constant γ such that for $t \geqq \gamma$,

(22) $$A(t - \gamma) \geqq B(t),$$

then

(23) $$\alpha_n^{(\omega)} \leqq \beta_n^{(\omega)} - n\gamma \leqq \beta_n - n\gamma \quad (n \geqq 1).$$

If $\omega(t) \equiv \infty$, then the hypothesis that γ is positive is useless, but, if $\gamma < 0$, the condition $t \geqq \gamma$ has to be replaced in each case by $t \geqq 0$.

Let us remark that, since $\alpha_n^{(\omega)} \leqq \alpha_n$, from

(24) $$\alpha_n \leqq \beta_n + n\gamma$$

there follows (20), and therefore by the Lemma, (21). But from (22) does not follow

$$\alpha_n \leqq \beta_n - n\gamma,$$

but precisely

$$\alpha_n^{(\omega)} \leqq \beta_n^{(\omega)} - n\gamma.$$

This Lemma has a certain Tauberian character which is difficult to state here. We shall nevertheless remark that it gives, in fact, a new point of view on Tauberian theorems.

Let us now pass to the proof of the Lemma.

We have, if $t \geqq \gamma$,

$$A(t) = \operatorname*{Max}_{n \leqq \omega(t)} (nt - \alpha_n^{(\omega)}) \geqq \operatorname*{Max}_{n \leqq \omega(t)} (nt - \beta_n - n\gamma)$$

$$= \operatorname*{Max}_{n \leqq \omega(t)} \big(n(t-\gamma) - \beta_n\big) \geqq \operatorname*{Max}_{n \leqq \omega(t-\gamma)} \big(n(t-\gamma) - \beta_n\big) = B(t-\gamma).$$

On the other hand, it follows from (20) that

$$\alpha_n^{(\omega)} = \operatorname*{\overline{Bd}}_{\omega(t) \geqq n} \big(nt - A(t)\big) = \operatorname*{\overline{Bd}}_{\omega(t-\gamma) \geqq n} \big(n(t-\gamma) - A(t-\gamma)\big)$$

$$= \operatorname*{\overline{Bd}}_{\omega(t-\gamma) \geqq n} \big(nt - A(t-\gamma)\big) - n\gamma \leqq \operatorname*{\overline{Bd}}_{\omega(t) \geqq n} \big(nt - B(t)\big) - n\gamma = \beta_n^{(\omega)} - n\gamma.$$

The last affirmation of the Lemma is evident by the preceding proof, since, if $\omega(t) \equiv \infty$, $\omega(t-\gamma) = \omega(t)$, for every $t \geqq 0$, if $\gamma \leqq 0$, and for $t \geqq \gamma$, if $\gamma > 0$.

REMARK: It is evident, by the proof of the Lemma III,[1] that, in its statement, the expressions "for $t \geqq \gamma$" (or "for $t \geqq 0$"), and "($n \geqq 1$)" may be respectively replaced by "*for t sufficiently large*" and "*n sufficiently large*," provided, of course, that both be replaced.

Returning to the general notions considered on page 13, consider now the set T_i. Such a set constitutes an interval, closed at the left-hand, and open at the right-hand. The

[1]And by the statement, of course, that for large values of t, the ω-trace depends only on the terms of the generatrix, with large indices, and that the terms of the ω-regularized sequence, for large values of n, depend only on the ω-trace for large values of t (see pages 19 and 22).

right-hand extremity was denoted by τ_i. Let us denote by τ_i^0 the left-hand extremity of T_i. Thus T_i is the interval

$$\tau_i^0 \leqq t < \tau_i.$$

T_i is, we recall, the set of values t such that the principal point P_{n_i} lies on D_t, t being any point of T_i. It is then obvious, by construction, that if $j > i$, no value of T_j can be smaller than any value of T_i. Thus $\tau_j^0 \geqq \tau_i$. On the other hand, since there is at least one principal point (a point of the sequence $\{P_{n_i}\}$) on every line D_t, every value $t \geqq 0$ belongs to at least one set T_i. Therefore

$$\tau_{i+1}^0 = \tau_i.$$

And the set which is the sum of all the sets T_i is equal to the set composed of all the points $t \geqq 0 : \Sigma T_i = [0, \infty)$.

We have seen that $A(t)$, the ω-trace of the sequence $\{\alpha_n\}$, is such that $-A(t)$ is the y-intercept of D_t. If t belongs to T_i, D_t passes through the point $P_{n_i} \equiv (n_i, \alpha_{n_i})$, and thus

$$A(t) = n_i t - \alpha_{n_i}, \ n_i \leqq \omega(t). \tag{25}$$

This proves also that, if $n(t)$ is the greatest principal index n_i satisfying (25), then $m(t) = n(t)$, since if there were an integer m such that $m > n(t)$ and such that

$$A(t) = mt - \alpha_m, \ m \leqq \omega(t),$$

the point P_m would also lie on D_t, and m would also be a principal index satisfying (25), contrary to the hypothesis that $n(t)$ is the greatest principal index having this property.

Thus $m(t)$ is a principal index, and precisely the greatest principal index n_i such that t belongs to the corresponding set T_i.

Suppose that $m(t)$ $(t \geqq 0)$ takes the values: $m_0 < m_1 < \cdots < m_k, \cdots$. To every k corresponds, as we have seen above, an $i = i(k)$ and a $p = p(k)$, such that $m_k = n_i < n_{i+1}, < \cdots < n_{i+p} = m_{k+1}$. The points of discontinuity for $m(t)$ being the points $t_1, t_2, \cdots$, in $[t_k, t_{k+1})$, $m(t) = m_k = m(t_k)$, and putting $t_0=0$, in $[0, t_1): m(t)=m_0=m(0)$. Therefore in $[t_k, t_{k+1})$

$$A(t)=m_k t-\alpha_{m_k}.$$

This shows that

$$t_{k+1}=\tau_{i(k)}=\tau_{i+1}^0=\tau_{i+1}=\cdots\tau_{i+p-1}^0=\tau_{i+p-1}=\tau_{i+p}^0 \qquad (i+p=i(k+1)). \tag{26}$$

This last formula will allow us to give an important statement of convergence which will be useful afterwards.

By formula (13) we have

$$A(t)=A(0)+\int_0^t m(\tau)d\tau+\int_0^t dV(\tau).$$

We shall denote the integral $\int_0^t dV(\tau)$ by $A_d(t)$ and call it the *discontinuous part* of the ω-trace function $A(t)$. The expression $A(0)+\int_0^t m(\tau)d\tau=A(t)-A_d(t)$ will be denoted by $A_c(t)$ and will be called the *continuous part* of $A(t)$,

$$A(t)=A_c(t)+A_d(t).$$

Both, the continuous and the discontinuous part, are increasing functions, the continuous part tends to infinity, the discontinuous part is not negative.

We can now prove the following Lemma:

LEMMA IV. *The following equality holds:*

$$\int_0^\infty A_c(t)e^{-t}dt=A(0)+m(0)+\sum_{i=1}^\infty (n_{i+1}-n_i)e^{-\tau_i},$$

where $\{n_i\}$ *is the sequence of principal indices of the sequence* $\{P_n\}$, *and where* $\tau_i=\overline{\mathrm{Bd}}\ T_i$.

We have

$$\int_0^x A_c(t)e^{-t}dt=A(0)\int_0^x e^{-t}dt+\int_0^x\left(\int_0^t m(\tau)d\tau\right)e^{-t}dt.$$

Integrating by parts the last integral we get

$$\int_0^x A_c(t)e^{-t}dt = A(0) - A(0)e^{-x} - e^{-x}\int_0^x m(t)dt + \int_0^x m(t)e^{-t}dt$$

$$= A(0) - e^{-x}A_c(x) + \int_0^x m(t)e^{-t}dt.$$

If the left-hand integral converges when $x \uparrow \infty$, there exists a sequence of quantities x, tending to infinity such that $e^{-x}A_c(x)$ tends to zero, and $\int_0^x m(t)e^{-t}dt + A(0)$ tends to $\int_0^\infty A_c(t)e^{-t}dt$. Conversely if $\int_0^x m(t)e^{-t}dt$ tends to a limit when $x \uparrow \infty$, the left-hand integral does also and $e^{-x}A_c(x)$ tends to zero. Thus

$$\int_0^\infty A_c(t)e^{-t}dt = A(0) + \int_0^\infty m(t)e^{-t}dt.$$

We prove, in exactly the same manner, by writing

$$\int_0^x m(t)e^{-t}dt = m(0) - m(x)e^{-x} + \int_0^x e^{-t}dm(t),$$

that

$$\int_0^\infty m(t)e^{-t}dt = m(0) + \int_0^\infty e^{-t}dm(t)$$

(and also, that the convergence of one of these integrals gives $m(x)e^{-x} \to 0$, when $x \uparrow \infty$). Therefore

$$\int_0^\infty A_c(t)e^{-t}dt = A(0) + m(0) + \int_0^\infty e^{-t}dm(t)$$

$$= A(0) + m(0) + \sum_{k=0}^\infty (m_{k+1} - m_k)e^{-t_{k+1}},$$

where the quantities t_k have the same meaning as on page 24, and thus, by formula (26),

$$\int_0^\infty A_c(t)e^{-t}dt = A(0) + m(0) + \sum_{i=1}^\infty (n_{i+1} - n_i)e^{-r_i}.$$

The value of each of these expressions, or of

$$\int_0^\infty m(t)e^{-t}dt, \quad \int_0^\infty e^{-t}dm(t),$$

may be infinity. But we have proved that they are all finite or infinite at the same time.

REMARK: We have seen, during the proof, that the convergence of one of these expressions (and therefore of all of them) involves

$$\lim_{x=\infty} A(x)e^{-x} = \lim_{x=\infty} m(x)e^{-x} = 0.$$

Let us now prove the following statements which will be useful in finding conditions for differentiability of classes.

LEMMA V.

$$(1) \quad \underline{\lim}\, \frac{m(t)}{t} \geqq \underline{\lim} \frac{A_c(t)}{t^2} \geqq \tfrac{1}{4}\, \underline{\lim}\, \frac{m(t)}{t},$$

$$(2) \quad 2\,\overline{\lim}\, \frac{\alpha_n^{(\omega)}}{n^2} \leqq \overline{\lim}\, \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n} = \overline{\lim}\, \frac{\alpha_{n+1}^{(\omega)} - \alpha_n}{n}.$$

If

$$\overline{\lim}\, \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n} \geqq 0$$

then

$$(3) \qquad \overline{\lim}\, \frac{t}{m(t)} = \overline{\lim} \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n}.$$

If

$$\underline{\lim}\, \frac{\omega(t)}{t} > 0,$$

the two conditions

$$(4) \qquad \underline{\lim}\, \frac{A(t)}{t^2} > 0,$$

$$(5) \qquad \overline{\lim}\, \frac{\alpha_n^{(\omega)}}{n^2} < \infty,$$

are equivalent.

Therefore if $\underline{\lim}\,\frac{\omega(t)}{t}>0$, *and if the discontinuous part of* $A(t)$ *is zero*: $A(t)=A_c(t)$, *then the conditions*

$$\underline{\lim}\,\frac{A(t)}{t^2}>0,$$

$$\underline{\lim}\,\frac{m(t)}{t}>0,$$

$$\overline{\lim}\,\frac{\alpha_n^{(\omega)}}{n^2}<\infty,$$

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}<\infty,$$

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n}{n}<\infty,$$

are equivalent.

The relation (1) results from the following:

$$A(0)+tm(t)\geqq A_c(t)=A(0)+\int_0^t m(\tau)d\tau\geqq A(0)+\int_{\frac{t}{2}}^t m(\tau)d\tau$$
$$\geqq A(0)+\frac{t}{2}\,m\left(\frac{t}{2}\right);$$

therefore

$$\underline{\lim}\,\frac{m(t)}{t}\geqq\underline{\lim}\,\frac{A_c(t)}{t^2}\geqq\tfrac{1}{2}\,\underline{\lim}\frac{m\left(\frac{t}{2}\right)}{t}$$
$$=\tfrac{1}{4}\,\underline{\lim}\,\frac{m\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)}=\tfrac{1}{4}\,\underline{\lim}\,\frac{m(t)}{t}.$$

The first part of the inequality (2) follows from these simple considerations: if the second expression in (2) is equal to k, then to every positive ϵ there corresponds a number n_ϵ such that for $n\geqq n_\epsilon$,

$$\alpha_{n+1}^{(\omega)}<\alpha_n^{(\omega)}+n(k+\epsilon);$$

thus

$$\alpha_{n+1}^{(\omega)}<\alpha_{n_0}^{(\omega)}+[n_0+(n_0+1)+\cdots+n](k+\epsilon)\leqq\alpha_{n_0}^{(\omega)}+\frac{n(n+1)}{2}(k+\epsilon),$$

and obviously the desired inequality follows.

In order to prove the inequality (3) let us consider, as on page 24, the values $m_0<m_1<\cdots<m_k<\cdots$, taken by $m(t)$ in the intervals $[0, t_1)$, $[t_1, t_2)$, $\cdots$, $[t_k, t_{k+1})$, $\cdots$, and consider an integer $n: m_k\leqq n<m_{k+1}$. It is seen, by (26), that, if $m_k=n_{i(k)}$, then

$$\frac{n}{\tau_{i(k)}}\geqq\frac{m_k}{t_{k+1}}=\frac{m(t_{k+1}-0)}{t_{k+1}}. \tag{27}$$

It is also seen, by (26), that $\tau_{i(k)}$ is the slope of the straight line passing through $(n_i, \alpha_{n_i}^{(\omega)})$ on which are situated all points $(n, \alpha_n^{(\omega)})$ with $n_i\leqq n<n_{i+1}$; the point $(n_{i+1}, \alpha_{n_{i+1}}^{(\omega)})$ is situated either on the same straight line or below it; therefore, since $n+1\leqq n_{i+1}$, we have

$$\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}\leqq\tau_{i(k)},$$

and, by (27),

$$\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}\leqq\frac{t_{k+1}}{m(t_{k+1}-0)}. \tag{28}$$

Now, if

$$\overline{\lim}\frac{t}{m(t)}=k,$$

we have, for sufficiently large values of t,

$$\frac{t}{m(t)}<k+\epsilon,$$

and by (28), we have, for large values of n,

$$\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}\leqq k+\epsilon.$$

Thus

$$\varlimsup_{n=\infty} \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n} \leqq \varlimsup_{t=\infty} \frac{t}{m(t)}.$$

It is also obvious that we have, in $[t_k, t_{k+1})$, the following inequality:

$$\frac{t}{m(t)} = \frac{t}{m_k} \leqq \frac{\tau_{i(k)}}{m_k} = \frac{\tau_{i(k)}}{m(\tau_{i(k)} - 0)}.$$

Thus

$$\varlimsup_{t=\infty} \frac{t}{m(t)} = \varlimsup_{k=\infty} \frac{\tau_{i(k)}}{m(\tau_{i(k)} - 0)}.$$

We know that, if the point $(m_k + 1, \alpha_{m_k+1}^{(\omega)})$ is below the straight line with slope $\tau_{i(k)}$, passing through $(m_k, \alpha_{m_k}^{(\omega)})$, then

$$m_k + 1 = m(\tau_{i(k)} - 0) + 1 = e^{\tau_{i(k)}},$$ [1]

which shows that, if, for a sequence $\{k_j\}$, we have

$$\varlimsup_{k=\infty} \frac{\tau_{i(k)}}{m(\tau_{i(k)} - 0)} = \lim_{j=\infty} \frac{\tau_{i(k_j)}}{m(\tau_{i(k_j)} - 0)},$$

this upper limit can be positive only if, for large values of j, the point $(m_{k_j} + 1, \alpha_{m_{k_j}+1}^{(\omega)})$ is on the straight line with slope $\tau_{i(k_j)}$ passing through the point $(m_{k_j}, \alpha_{m_{k_j}}^{(\omega)})$. Therefore, if this upper limit is not zero, we have

$$\varlimsup_{n=\infty} \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n} \geqq \lim_{j=\infty} \frac{\alpha_{m_{k_j}+1}^{(\omega)} - \alpha_{m_{k_j}}^{(\omega)}}{m_{k_j}} = \lim_{j=\infty} \frac{\tau_{i(k_j)}}{m(\tau_{i(k_j)} - 0)}$$
$$= \varlimsup_{t=\infty} \frac{t}{m(t)}.$$

In the case when the last expression is zero, the inequality

$$\varlimsup_{n=\infty} \frac{\alpha_{n+1}^{(\omega)} - \alpha_n^{(\omega)}}{n} \geqq \varlimsup_{t=\infty} \frac{t}{m(t)}$$

is obvious, by the hypothesis on the first expression.

[1]We recall that the point $(m_k + 1, \alpha_{m_k+1}^{(\omega)})$ can be either on the described line, or below it, but can never be above it.

We have thus proved that

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}=\overline{\lim}\,\frac{t}{m(t)}.$$

Let us now prove that

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n}{n}=\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}.$$

It is obvious that

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n}{n}\leqq\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n},$$

since $\alpha_n\geqq\alpha_n{}^{(\omega)}$. In order to prove that

$$\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n}{n}\geqq\overline{\lim}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n},$$

we remark that, if $n_i\leqq n<n+1\leqq n_{i+1}$, where n_i is a principal index, then

$$\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}\leqq\alpha_{n_i+1}^{(\omega)}-\alpha_{n_i}^{(\omega)}=\alpha_{n_i+1}^{(\omega)}-\alpha_{n_i}.$$

Indeed, the last part of this relation, the equality, is obvious, since for a principal index $\alpha_{n_i}=\alpha_{n_i}^{(\omega)}$. The first part is also evident, since, if $n+1<n_{i+1}$, or if $n+1=n_{i+1}$, the index n_{i+1} being not an index of discontinuity, the points

$$(n_i,\ \alpha_{n_i}^{(\omega)}),\ (n_i+1,\ \alpha_{n_i+1}^{(\omega)}),\ (n,\ \alpha_n{}^{(\omega)}),\ (n+1,\ \alpha_{n+1}^{(\omega)})$$

are on the same straight line passing through $(n_i,\ \alpha_{n_i}^{(\omega)})$ and with slope τ_i, and, if n_{i+1} is an index of discontinuity, and $n+1=n_{i+1}$, then the point $(n+1,\ \alpha_{n+1}^{(\omega)})\equiv(n_{i+1},\ \alpha_{n_i+1}^{(\omega)})$ is below this line, the other points being on this line. Thus, in each case, the desired inequality is satisfied. We have then

$$\overline{\lim_{n=\infty}}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n^{(\omega)}}{n}\leqq\overline{\lim_{i=\infty}}\,\frac{\alpha_{n_i+1}^{(\omega)}-\alpha_{n_i}^{(\omega)}}{n_i}=\overline{\lim_{i=\infty}}\,\frac{\alpha_{n_i+1}^{(\omega)}-\alpha_{n_i}}{n_i}\leqq\overline{\lim_{n=\infty}}\,\frac{\alpha_{n+1}^{(\omega)}-\alpha_n}{n}.$$

And the inequality (2) is completely proved.

Let us now pass to (4) and (5).

By the conditions imposed on $\omega(t)$ there exists a positive constant k such that

(29) $$\omega(t)>kt.$$

If (3) holds, there exists a positive constant k_1, which may be supposed smaller than $\frac{k}{2}$, such that for large values of t

$$A(t) > k_1 t^2,$$

and therefore, for large values of n,

$$\alpha_n^{(\omega)} = \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - A(t)) \leqq \overline{\mathrm{Bd}}_{\omega(t) \geqq n} (nt - k_1 t^2).$$

Since $nt - k_1 t^2$ takes its maximum at $t = \frac{n}{2k_1}$, and, by (29), $\omega\left(\frac{n}{2k_1}\right) \geqq \omega\left(\frac{n}{k}\right) > n$ (for sufficiently large values of n), the least upper bound is equal to this maximum, i.e., to $\frac{n^2}{4k_1}$, which proves that from (4) follows (5).

Now suppose that (5) is satisfied. We may suppose then that for large $n : n \geqq n_0$,

$$\alpha_n{}^{(\omega)} < k_1 n^2,$$

where the constant k_1 is such that

$$\frac{t}{2k_1} < \omega(t) - 1,$$

for large values of t (by (29)). But then, for large values of t,
$A(t) = \mathrm{Max}_{n \leqq \omega(t)} (nt - \alpha_n{}^{(\omega)}) \geqq \mathrm{Max}_{n_0 \leqq n \leqq \omega(t)} (nt - k_1 n^2) = \mathrm{Max}_{n \geqq n_0} (nt - k_1 n^2)$,
since the last expression takes its maximum at $\left[\frac{t}{2k_1}\right]$ or at $\left[\frac{t}{2k_1}\right] + 1$, both of which are smaller than $\omega(t)$, and greater than n_0, for large values of t.

Thus, for large values of t,

$$A(t) \geqq \left[\frac{t}{2k_1}\right] t - k_1 \left[\frac{t}{2k_1}\right]^2 \geqq \left(\frac{t}{2k_1} - 1\right) t - k_1 \left(\frac{t}{2k_1}\right)^2 = \frac{t^2}{4k_1} - t,$$

which gives (4) immediately.

6. EXPONENTIAL AND CONVEX REGULARIZATION

We shall have to deal, in these lectures, with two types of regularization: one with respect to the function $\omega(t)=e^t$, which is of the type (1) (see page 12), the other, with respect to the function $\omega(t)$, identically equal to $+\infty$ (type (3).)

The first case, $\omega(t)=e^t$ is, by far, the more important one, and generally the theory we have been concerned with in the last paragraph is of interest mostly in view of the regularization with respect to a function of the type (1) or (2), since regularization with respect to $\omega(t)\equiv\infty$ could be treated in a much simpler manner and is nothing but the rectification by the well known *polygon of Newton* (**20**).

If $\omega(t)=e^t$, we shall write instead of $\alpha_n^{(\omega)}$, $P_n^{(\omega)}$, ω-base, ω-trace, ω-generatrix, respectively, $\alpha_n^{(\text{exp.})}$, $P_n^{(\text{exp.})}$, *exp.-base, exp.-trace, exp.-generatrix* (read "*exponential-base*" etc . . .). The regularization with respect to e^t will be called *exponential regularization*, the sequence $\{\alpha_n^{(\text{exp.})}\}$—the *exponentially regularized sequence of* $\{\alpha_n\}$.

Let $\{M_n\}$ be a sequence of positive numbers, an infinity of which are finite, M_1 being also finite. Put $\alpha_n=\log M_n$ $(n\geqq 1)$, and let us write $M_n^o=e^{\alpha_n^{(\text{exp})}}$. The sequence $\{M_n^o\}$ so defined will be called the *exponentially regularized sequence of* $\{M_n\}$ *regularized by means of logarithms.*

For every $r\geqq 1$ let us put

$$S(r)=\operatorname*{Max}_{n\leqq r}\frac{r^n}{M_n}.$$

If we write $r=e^t$, we have

$$A(t)=\log S(e^t)=\operatorname*{Max}_{n\leqq r}(nt-\log M_n)=\operatorname*{Max}_{n\leqq e^t}(nt-\alpha_n).$$

Thus $A(t)$ is the exp-trace of $\{\alpha_n\}\equiv\{\log M_n\}$. Therefore by the Lemmas I and II,

$$\log M_n^o=\alpha_n^{(\text{exp})}=\overline{\operatorname*{Bd}_{e^t\geqq n}}\,(nt-A(t))=\overline{\operatorname*{Bd}_{r\geqq n}}\,(n\log r-\log S(r)),$$

$A(t)=\log S(r)=\underset{n\leqq e^t}{\mathrm{Max}}(nt-\alpha_n^{(\mathrm{exp})})=\underset{n\leqq r}{\mathrm{Max}}\,(n\log r-\log M_n^o).$

That is to say,

$$M_n^o=\underset{r\geqq n}{\overline{\mathrm{Bd}}}\,\frac{r^n}{S(r)},$$

$$S(r)=\underset{n\leqq r}{\mathrm{Max}}\,\frac{r^n}{M_n^o}.$$

The following statements are simple translations of corresponding statements in the general theory, expressed in the case $\omega(t)=e^t$, and where the quantities t, α_n, $\alpha_n^{(\mathrm{exp})}$ are replaced by $r=e^t$, $M_n=e^{\alpha_n}$, $M_n^o=e^{\alpha_n^{(\mathrm{exp})}}$ etc.

If the two sequences $\{M_n\}$ and $\{N_n\}$ are such that $\dfrac{M_n}{N_n}=k$ (constant), then $\dfrac{M_n^o}{N_n^o}=k$.

LEMMA III[1]. *If $\{M_n\}$ and $\{N_n\}$ are two positive sequences, such that there exists a constant $q\geqq 1$ such that*

(30) $$M_n^o\leqq q^nN_n\ (n\geqq 1),$$

then, denoting by

$$S_M(r)=\underset{n\leqq r}{\mathrm{Max}}\,\frac{r^n}{M_n},$$

$$S_N(r)=\underset{n\leqq r}{\mathrm{Max}}\,\frac{r^n}{N_n},$$

we have for every $r\geqq q$

(31) $$S_M(r)\geqq S_N\left(\frac{r}{q}\right).$$

If there exists a constant $q\geqq 1$, such that for every $r\geqq q$

(32) $$S_M\left(\frac{r}{q}\right)\geqq S_N(r),$$

then

(33) $$M_n^o\leqq\frac{N_n^o}{q^n}\leqq\frac{N_n}{q^n},$$

where $\{M_n^o\}$ *and* $\{N_n^o\}$ *denote, respectively, the exponentially regularized sequences of* $\{M_n\}$ *and* $\{N_n\}$, *regularized by means of logarithms.*

REMARK. If (30) holds for large values of n, (31) holds for large values of r. If (32) holds for large values of r, (33) holds for large values of n.

For example, let us put

$$N_n = \alpha^n n! \ (n \geqq 1) \ (\alpha \ constant > 1).$$

We then have

$$S_N(r) = \operatorname*{Max}_{n \leqq r} \frac{r^n}{\alpha^n n!} = \frac{\left(\frac{r}{\alpha}\right)^{\left[\frac{r}{\alpha}\right]}}{\left[\frac{r}{\alpha}\right]!}.$$

That is to say,

$$e^{\frac{\alpha(r)r}{\alpha}} \leqq S_N(r) < e^{\frac{r}{\alpha}}, \tag{34}$$

with $\alpha(r) \to 1$, when $r \to \infty$.

We have also

$$N_n \geqq N_n^o = \overline{\operatorname*{Bd}}_{r \geqq n} \frac{r^n}{S_N(r)} = \overline{\operatorname*{Bd}}_{r \geqq n} \left[\frac{r}{\alpha}\right]! \ \alpha^{\left[\frac{r}{\alpha}\right]} r^{n-\left[\frac{r}{\alpha}\right]}$$

$$\geqq \alpha^n n! \ (\alpha n)^{n-n} = \alpha^n n! = N_n.$$

In other words, in this case, the exponentially regularized sequence $\{N_n^o\}$, regularized by means of logarithms, is the sequence itself.

The Lemma III(I), or rather the remark which follows this Lemma, may be translated, therefore, in the following manner (writing $S(r)$ instead of $S_M(r)$, and putting $N_n = n!$):

LEMMA III(II). *Let* $\{M_n\}$ *be a sequence of positive numbers; let us put*

$$S(r) = \operatorname*{Max}_{n \leqq r} \frac{r^n}{M_n}.$$

The two conditions:

(1) *There exists a positive constant* μ, *such that for large values of* r

$$S(r) > e^{\mu r},$$

(2) $$(M_n^o)^{1/n} = O(n),\text{[1]}$$

are equivalent.

That is to say, from (1) follows (2), and conversely.

With the notation, $N_n = \alpha^n n!$, from (34), and (1) it follows that for large values of r

$$S(r) > S_N(qr),$$

where $q = \mu\alpha > 1$, if α is chosen such that $\alpha\mu > 1$. That is to say,

$$S\left(\frac{r}{q}\right) > S_N(r),$$

for large values of r, and by Lemma III(I), for large values of n,

$$M_n^o \leqq \frac{N_n}{q_n} = \left(\frac{\alpha}{q}\right)^n n!.$$

Thus (2) holds.

If now (2) holds, we have for $q > 1$ and sufficiently large

$$M_n^o < q^n n!,$$

and with the same notations as above, we have (with $\alpha = 1$, i.e., $N_n = n!$) by the Lemma III(I)

$$S(r) \geqq S_N\left(\frac{r}{q}\right),$$

and by (34) for large r with a suitable $\mu > 0$,

$$S(r) > e^{\mu r}.$$

The Lemma is thus proved.

[1]In Landau's notation, if $\varphi(n)$ and $\psi(n)$ are positive, $\varphi(n) = O(\psi(n))$ means that $\frac{\varphi(n)}{\psi(n)}$ is bounded $\left(\frac{\varphi(n)}{\psi(n)} < M\right)$.

Let us now translate the formula (13). We denote by $M(r)$ the greatest integer n such that $1 \leqq n \leqq r$ and such that

$$S(r) = \frac{r^n}{M_n}.$$

We have thus

$$S(r) = \frac{r^{M(r)}}{M_{M(r)}},$$

and by the notations employed in (13): $M(r) = m(\log r)$. Thus (13) becomes, if $r > r' \geqq 1$,

$$\text{(35)} \qquad \log S(r) = \log S(r') + \int_{r'}^{r} \frac{M(x)}{x}\, dx + \int_{r'}^{r} dW(x),$$

where $W(x)$ is constant in every interval $[r_n, r_{n+1})$, this function being discontinuous at the points r_n, which are the points of discontinuity of $S(r)$ (with the notations of page 18: $r_n = e^{t_n}$). We have $dW(r_n) = W(r_n) - W(r_n - 0) > 0$. We recall also that $S(r)$ and $M(r)$ are functions increasing to infinity, and that $S(r) = S(r+0)$, $M(r) = M(r+0)$.

Let us now pass to the simple case of convex regularization. If $\omega(t)$ is identically equal to ∞, we shall write instead of $\alpha_n^{(\omega)}$, $P_n^{(\omega)}$, ω-base, etc., $\alpha_n^{(\infty)}$, $P_n^{(\infty)}$, ∞-base, etc.

In this case there are no indices of discontinuity, and the ∞-base of a sequence of points $\{P_n\}$ (in the case $\omega(t) \equiv \infty$, we suppose always $\lim \frac{\alpha_n}{n} = \infty$) is a continuous convex polygonal line, with the convexity turned toward y negative. This is the reason why we call this kind of regularization convex regularization, and $\{\alpha_n^{(\infty)}\}$, $\{P_n^{(\infty)}\}$, the convex regularized sequences of $\{\alpha_n\}$ and $\{P_n\}$.

Let $\{M_n\}$ be a sequence of positive numbers such that $\lim (M_n)^{1/n} = \infty$ and put $\alpha_n = \log M_n$. Since $\lim \frac{\alpha_n}{n} = \infty$, the sequence $\{\alpha_n\}$ can be regularized convexly. Let us write $M_n^c = e^{\alpha_n^{(\infty)}}$. The sequence $\{M_n^c\}$, so defined, will be called the *convex regularized sequence of* $\{M_n\}$ *regularized by means of logarithms.*

For every $r \geqq 1$, let us put

$$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n}.$$

This maximum exists since $\lim (M_n)^{1/n} = \infty$. Writing $r = e^t$, we have

$$A(t) = \log T(r) = \operatorname*{Max}_{n \geqq 1} (nt - \log M_n) = \operatorname*{Max}_{n \geqq 1} (nt - \alpha_n).$$

Thus $A(t)$ is the ∞-trace of $\{\alpha_n\} \equiv \{\log M_n\}$ and by Lemmas I and II

$$\log M_n^c = \alpha_n^{(\infty)} = \overline{\operatorname*{Bd}_{t \geqq 0}} \,(nt - A(t)) = \overline{\operatorname*{Bd}_{r \geqq 1}} \,(n \log r - \log T(r)),$$

$$A(t) = \log T(r) = \operatorname*{Max}_{n \geqq 1} (nt - \alpha_n^{(\infty)}) = \operatorname*{Max}_{n \geqq 1} (n \log r - \log M_n^c).$$

That is to say,

$$M_n^c = \overline{\operatorname*{Bd}_{r \geqq 1}} \frac{r^n}{T(r)} = \operatorname*{Max}_{r \geqq 1} \frac{r^n}{T(r)},[1]$$

$$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n^c}.$$

Lemma III becomes here (taking care of the last statement in this Lemma) with the notations

$$T_M(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n}, \quad T_N(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{N_n}:$$

LEMMA III[(III)]. *If there exists a constant* $q > 1$, *such that*

(36) $$M_n^c \leqq q^n N_n (n \geqq 1),$$

then, for $r \geqq$ *max* $(q, 1)$:

(37) $$T_M(r) \geqq T_N\left(\frac{r}{q}\right).$$

[1]It is obvious that $T(r)$ is continuous and that for every $n \geqq 1, \frac{r^n}{T(r)} \to 0$ when $r \to \infty$. Therefore $\overline{\operatorname*{Bd}_{r \geqq 1}}$ may be replaced, in this case, by $\operatorname*{Max}_{r \geqq 1}$.

And conversely, if (37) *is valid for* $r \geqq max\ (q, 1)$, *there follows* (36) *for* $n \geqq 1$.

If (36) holds for n large, (37) holds for r large, and conversely.

By a proof analogous to that of III$^{(II)}$ we can establish

LEMMA III$^{(IV)}$. *The two conditions*

(1) *There exists a positive constant* μ *such that for large values of* r

$$T(r) > e^{\mu r},$$

$$(M_n^c)^{1/n} = O(n), \tag{2}$$

are equivalent.

Let us denote by $N(r)$ the greatest positive integer n such that

$$T(r) = \frac{r^n}{M_n}.$$

We have then

$$T(r) = \frac{r^{N(r)}}{M_{N(r)}}.$$

Here also $N(r) = m(\log r)$ (see page 37). And, since $T(r)$ is a continuous function, (13) is translated into the simple form

$$\log\ T(r) = \log\ T(r') + \int_{r'}^{r} \frac{N(x)}{x}\, dx, \tag{38}$$

with $N(r) \uparrow \infty$, and $N(r+0) = N(r)$.

Such an integral valued function $N(x)$, and the corresponding formula (38) were given, in the theory of entire functions, by Valiron (**20**), in dealing with Newton's polygon.

We shall use later Lemma IV in the case of convex regularization; we need therefore to translate its statement in this case. Since $T(r)$ is continuous $A_c(t) = \log\ T(r)$ $(r = e^t)$. Here $\tau_i = \alpha_{n_i+1}^{(\infty)} - \alpha_{n_i}^{(\infty)} = \alpha_{n_i+2}^{(\infty)} - \alpha_{n_i+1}^{(\infty)} \cdots = \alpha_{n_{i+1}}^{(\infty)} - \alpha_{n_{i+1}-1}^{(\infty)}$.

That is to say,

$$e^{-r_i}=\frac{M^c_{n_i}}{M^c_{n_i+1}}=\frac{M^c_{n_i+1}}{M^c_{n_i+2}}=\frac{M^c_{n_{i+1}-1}}{M^c_{n_{i+1}}}$$

and

$$(n_{i+1}-n_i)e^{-r_i}=\sum_{n=n_i}^{n_{i+1}-1}\frac{M^c_n}{M^c_{n+1}}.$$

Therefore Lemma IV becomes

LEMMA IV[(1)]

$$\int_1^\infty\frac{\log T(r)}{r^2}\,dr=\log T(1)+N(1)+\sum_{n=n_1}^{\infty}\frac{M^c_n}{M^c_{n+1}}=$$
$$\log T(1)+1+\sum_{n=1}^{\infty}\frac{M^c_n}{M^c_{n+1}}.$$

Thus the integral and the series, in this expression, converge or diverge together. The remarks which follow Lemma IV may be translated in the following manner: of the convergence of either the integral, or the series involved in Lemma IV[(1)], it follows that

$$\lim_{r=\infty}\frac{\log T(r)}{r}=\lim_{r=\infty}\frac{N(r)}{r}=0.$$

The convergence (or divergence) of the integral (and the series) involved in Lemma IV[(1)], is equivalent to that of the integrals

$$\int_1^\infty\frac{N(r)}{r^2}\,dr,\quad \int_1^\infty\frac{dN(r)}{r}.$$

7. SOLUTION OF THE PROBLEM OF ANALYTICITY OF A CLASS

We shall give in this paragraph the complete solution of the problem of analyticity of a class, proposed by Carleman in 1926(**2**) and solved by the present author in 1935(**9**). The solution is given by the following fundamental theorem.

THEOREM II. *A necessary and sufficient condition for the analyticity of a class* $C_{\{M_n\}}$ *of i.d. functions, defined in an interval* $I=[a, b]$, *is that*

$$(M_n^o)^{1/n}=O(n), \tag{39}$$

where $\{M_n^o\}$ *is the exponentially regularized sequence of* $\{M_n\}$, *regularized by means of logarithms.*

This means that, if (39) holds, then every function of $C_{\{M_n\}}$, defined in $[a, b]$, is analytic; and, if (39) does not hold, there is at least one function of $C_{\{M_n\}}$ which is not analytic in $[a, b]$.

Before we prove this fundamental theorem we need to prove some other statements. Let us begin by

THEOREM III. *A necessary and sufficient condition in order that a function, i.d. for* $-\infty<x<\infty$, *and periodic (with period* 2π),

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

be analytic, is that there exist a positive constant γ *such that for large values of n*

$$\begin{aligned} |a_n| &< e^{-\gamma n} \\ |b_n| &< e^{-\gamma n}. \end{aligned} \tag{40}$$

We have for $n\geqq 1$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx.$$

Integration by parts, repeated p times yields

$$a_n=\pm\frac{1}{n^p}\frac{1}{\pi}\int_0^{2\pi} f^{(p)}(x)\begin{matrix}\sin nx\\ \text{or}\\ \cos nx\end{matrix}\,dx.$$

But, since $f(x)$ is analytic in $[0, 2\pi]$, we have by Theorem I

$$|f^{(p)}(x)|<k^p p! \quad [p\geqq 1,\ 0\leqq x\leqq 2\pi]$$

where k is a constant. Hence

$$|a_n|<\frac{1}{\pi}\frac{2\pi}{n^p}k^p p!=2\left(\frac{k}{n}\right)^p p!.$$

Since this is true for every $p \geqq 1$, let us put $p = \left[\frac{n}{k}\right]$:

$$|a_n| \leqq \frac{2\left[\frac{n}{k}\right]!}{\left[\frac{n}{k}\right]^{\left[\frac{n}{k}\right]}}.$$

And by Stirling's formula, for large values of n,

$$|a_n| < e^{-\gamma n} \qquad (\gamma > 0).$$

We prove, in the same manner, that for large n,

$$|b_n| < e^{-\gamma n}.$$

Suppose now that (40) holds for $n \geqq n_0$. In the strip $|y| < a$, the functions $\cos nz$ and $\sin nz$ satisfy the inequalities $(z = x + iy)$:

$$|\cos nz| = \left|\frac{e^{inz} + e^{-inz}}{2}\right| \leqq \frac{e^{ny} + e^{-ny}}{2} \leqq e^{na},$$

$$|\sin nz| = \left|\frac{e^{inz} - e^{-inz}}{2i}\right| \leqq \frac{e^{ny} + e^{-ny}}{2} \leqq e^{na}.$$

And, if $a < \gamma$, it follows from (40) that in $|y| \leqq a$

$$\sum_{n=n_0}^{\infty} |a_n \cos nz + b_n \sin nz| \leqq 2 \sum_{n=n_0}^{\infty} e^{-(\gamma - a)n}.$$

The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

converges uniformly in $|y| \leqq a$, and therefore represents a holomorphic function in $|y| < a$. The Lemma is thus proved.

Next we shall introduce Tchebytcheff polynomials and give an inequality characterizing the growth of their successive derivatives.

The Tchebytcheff polynomial of degree n is given by

$$T_n(z) = \cos n(\text{arc} \cos z).$$

Putting $z = \cos\theta$, we see that if z is real and $|z| \leqq 1$,

$$T_n(z) = \cos n\theta = R(\cos\theta + i \sin\theta)^n = R(\cos\theta + i\sqrt{1-\cos^2\theta})^n$$
$$= R(z + i\sqrt{1-z^2})^n = R(z+\sqrt{z^2-1})^n = \tfrac{1}{2}[(z+\sqrt{z^2+1})^n$$
$$+ (z-\sqrt{z^2-1})^n],$$ [1]

and it is seen immediately that $T_n(z)$ is a polynomial in z of degree n.

We shall denote by C_R the ellipse with foci at the real points -1 and $+1$, the equation of which is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with $a+b=R$. We have then $a=\frac{1}{2}\left(R+\frac{1}{R}\right)$, $b=\frac{1}{2}\left(R-\frac{1}{R}\right)$.

And, if z is on C_R, we have $x=a\cos\varphi$, $y=b\sin\varphi$, that is to say,

(41) $$z=\tfrac{1}{2}(Re^{i\varphi}+R^{-1}e^{-i\varphi}).$$

If we put $z=\cos\theta$, with $\theta=\theta_1+i\theta_2$, we get

$$z=\frac{e^{i\theta_1-\theta_2}+e^{-i\theta_1+\theta_2}}{2}.$$

This proves that, if z is on C_R, $\theta_1=\varphi$, $\theta_2=-\log R$, and we have then, if z is on C_R,

$$T_n(z)=\cos n(\theta_1+i\theta_2)=\tfrac{1}{2}(R^n e^{in\varphi}+R^{-n}e^{-in\varphi}),$$

and

(42) $$|T_n(z)| \leqq R^n.$$

In order to give the growth of the maxima of $T_n{}^{(p)}$ in an interval contained in $[-1, +1]$, we note that the following differential equation holds:

(43) $$(x^2-1)T_n^{(p+2)}(x)+(2p+1)xT_n^{(p+1)}(x)=[n^2-p^2]T_n^{(p)}(x).$$

With the same notations as before, we have indeed

$$T_n(z)=\cos n\theta,$$
$$T_n'(z)=\frac{n\sin n\theta}{\sin\theta},$$
$$T_n''(z)=-n\frac{n\cos n\theta\sin\theta-\cos\theta\sin n\theta}{\sin^3\theta},$$

[1] $R\zeta$ denotes the real part of ζ.

which proves immediately, by substitution, that (43) holds for $p=0$. By differentiating (43) one sees immediately that if (43) holds for $p=q$, then it holds also for $p=q+1$. Thus (43) holds for every p.

We shall prove that if $|x|<1$, then

$$|T_n^{(p)}(x)| \leqq \frac{3^p n^p}{(1-x^2)^p}. \tag{44}$$

n being given, we suppose that (44) holds for $p=q$ and $p=q+1$ and prove that it holds for $p=q+2$.
By (43) we have

$$|T_n^{(p+2)}(x)| \leqq \frac{(2p+1)\,|T_n^{(p+1)}(x)|}{1-x^2}+\frac{n^2\,|T_n^{(p)}(x)|}{1-x^2},$$

and using (44) for $p=q$ and $p=q+1$, we have for $q+2 \leqq n$,

$$|T_n^{(q+2)}(x)| \leqq \frac{2n\cdot 3^{q+1}n^{q+1}}{(1-x^2)^{q+2}}+\frac{3^q n^{q+2}}{(1-x^2)^{q+1}}$$
$$\leqq \frac{3^{q+2}n^{q+2}}{(1-x^2)^{q+2}}.$$

But it is immediately seen that (44) holds for $p=0$ and $p=1$, by verification. Thus (44) holds for every p.

We shall now prove the following Lemma which will be very useful:

LEMMA VI. *Every function $f(x)$ belonging, in $[-1, +1]$ to $C_{\{M_n\}}$, can be expanded in a series of the form*

$$f(x)=\sum_0^\infty a_n T_n(x),$$

where the a_n satisfy the inequalities

$$|a_n| < \frac{1}{S\left(\dfrac{n}{A}\right)},$$

where

$$S(r)=\underset{n \leqq r}{\mathrm{Max}}\ \frac{r^n}{M_n},$$

and where A is a constant.

In putting $x=\cos\theta$, $f(x)=f(\cos\theta)=F(\theta)$ is an i. d., periodic, even function of θ in $[0, 2\pi]$. Therefore

$$F(\theta)=\sum_0^\infty a_n \cos n\theta,$$

and

$f(x)=F(x)=\Sigma a_n \cos n\theta=\Sigma a_n \cos n(\text{arc}\cos x)=\Sigma a_n T_n(x)$.

Such an expansion is also valid for every derivative $f^{(q)}(x)$. Let us then write

$$f^{(q)}(x)=\Sigma a_n^{(q)} T_n(x).$$

We have

$$f^{(q+1)}(x)\sin\theta=\sin\theta\Sigma a_n^{(q+1)}\cos n\theta=\sin\theta \frac{d}{d\theta}[f^{(q)}(\cos\theta)]\cdot\frac{d\theta}{dx}$$

$$=-\frac{d}{d\theta}[f^{(q)}(\cos\theta)],$$

where $f^{(q)}(\cos\theta)$ denotes the qth derivative with respect to the variable $x=\cos\theta$. But on the other hand

$$\frac{d}{d\theta}[f^{(q)}(\cos\theta)]=\Sigma(a_n^{(q)}\cos n\theta)'_\theta=-\Sigma n a_n^{(q)}\sin n\theta.$$

This gives

$$\sin\theta\sum a_n^{(q+1)}\cos n\theta=\sum n a_n^{(q)}\sin n\theta.$$

Using then the trigonometric identity,

$$\sin\theta\cos n\theta=\tfrac{1}{2}[\sin(n+1)\theta-\sin(n-1)\theta],$$

the above equality can be written

$$\sum\frac{a_n^{(q+1)}}{2}[\sin(n+1)\theta-\sin(n-1)\theta]=\sum n a_n^{(q)}\sin n\theta,$$

or

$$a_0^{(q+1)}\frac{\sin\theta}{2}+\tfrac{1}{2}\sum_{n=1}^\infty\left(a_{n-1}^{(q+1)}-a_{n+1}^{(q+1)}\right)\sin n\theta=\sum_1^\infty n a_n^{(q)}\sin n\theta.$$

Therefore

$$a_n^{(q)}=\frac{a_{n-1}^{(q+1)}-a_{n+1}^{(q+1)}}{2n}. \tag{45}$$

From

$$f^{(q)}(\cos\theta) = \sum a_n^{(q)}\cos n\theta$$

follows, for $n \geqq 1$,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f^{(q)}(\cos\theta)\cos n\theta d\theta.$$

Integrating by parts we get

$$a_n^{(q)} = -\frac{1}{n\pi}\int_0^{2\pi}\frac{df^{(q)}(\cos\theta)}{d\theta}\sin n\theta d\theta.$$

But since $f(x)$ belongs to $C_{\{M_n\}}$, in $[-1, +1]$,

$$\left|\frac{df^{(q)}(\cos\theta)}{d\theta}\right| = |f^{(q+1)}(x)\sin\theta| < k^{q+1}M_{q+1}(q \geqq 1),$$

and

$$|a_n^{(q)}| < \frac{2k^{q+1}M_{q+1}}{n}.$$

This gives by (44), substituting $n-p+1$ for n, and $p-1$ for q,

$$|a_{n-p+1}^{(p-1)}| \leqq \frac{|a_{n-p}^{(p)}| + |a_{n-p+2}^{(p)}|}{2(n-p+1)}$$

$$< \frac{k^{p+1}M_{p+1}}{n-p+1}\left(\frac{1}{n-p} + \frac{1}{n-p+2}\right) < \frac{2k^{p+1}M_{p+1}}{(n-p+1)(n-p)}.$$

Passing now from $p-1$ to $p-2$, and repeating this operation $p-1$ times, since $a_n^0 = a_n$, we shall have

$$|a_{n-p+2}^{(p-2)}| \leqq \frac{|a_{n-p+1}^{(p-1)}| + |a_{n-p+3}^{(p-1)}|}{2(n-p+2)}$$

$$< \frac{k^{p+1}M_{p+1}}{n-p+2}\left(\frac{1}{(n-p)(n-p+1)} + \frac{1}{(n-p+2)(n-p+3)}\right)$$

$$< \frac{2k^{p+1}M_{p+1}}{(n-p)(n-p+1)(n-p+2)}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$|a_n^0| = |a_n| < 2\frac{k^{p+1}M_{p+1}}{(n-p)(n-p+1)\cdots n},$$

where $n \geqq p+1$.

But if $0 \leqq l \leqq p$,

$$n-l \geqq \frac{p+1-l}{p+1} n,$$

and therefore there exists a numerical constant C such that

$$|a_n| < \frac{(kC)^{p+1} M_{p+1}}{n^{p+1}} (1 \leqq p \leqq n-1), \tag{46}$$

or

$$|a_n| < \frac{B^q M_q}{n^q} (2 \leqq q \leqq n),$$

where B is a constant.

It is also evident that (since $\Sigma a_n \cos n\theta$ is an i. d. function)

$$|a_n| < \frac{C_1}{n} (n \geqq 1),$$

where C_1 is a constant. Thus

$$|a_n| \leqq \frac{A^q M_q}{n^q} (1 \leqq q \leqq n,\ n \geqq 1),$$

where A is a constant, which, we can of course suppose, is greater than 1. We can then write

$$|a_n| \leqq \min_{q \leqq n} \frac{A^q M_q}{n^q} \leqq \min_{q \leqq \frac{n}{A}} \frac{A^q M_q}{n^q}$$

$$= \frac{1}{\operatorname{Max}\limits_{q \leqq \frac{n}{A}} \frac{\left(\frac{n}{A}\right)^q}{M_q}} = \frac{1}{S\left(\frac{n}{A}\right)},$$

and the Lemma is proved.

We shall now prove an important theorem which is another form of the fundamental Theorem II.

THEOREM IV. *Let $C_{\{M_n\}}$ be a class of i. d. functions defined in $[a, b]$. A necessary and sufficient condition for this class* *to be analytic in* $[a, b]$ *is that there exist a positive constant* μ *such that, for large values of* r

$$S(r) > e^{\mu r},$$

where

$$S(r) = \operatorname*{Max}_{n \leqq r} \frac{r^n}{M_n}.$$

Let us remark that, if all the functions of a class $C_{\{M_n\}}$, defined in an interval $[a, b]$, are analytic, then also all the functions of this class defined in any interval $[a', b']$ are analytic. This is seen immediately, since if $f(x)$ belongs to $C_{\{M_n\}}$ in $[a', b']$, then by a suitable choice of α and β, the function $f(\alpha y+\beta)=F(y)$ belongs to $C_{\{M_n\}}$ in $[a, b]$, but if $C_{\{M_n\}}$ is analytic, when defined in $[a, b]$, $F(y)$ is analytic, therefore $f(x)$ is analytic in $[a', b']$.

Thus, in the proof of the necessity of the conditions for analyticity we shall choose the interval $[-\frac{1}{2}, \frac{1}{2}]$, and for the sufficiency the interval $[-1, +1]$.

We shall prove that, if all the functions of $C_{\{M_n\}}$ defined in $[-\frac{1}{2}, \frac{1}{2}]$ are analytic, then there exists a positive constant μ such that $S(r) > e^{\mu r}$, for large values of r.

Consider, in the interval $[-\frac{1}{2}, \frac{1}{2}]$, the function

$$\varphi(x) = \sum_1^\infty C_n T_{3n}(x), \tag{47}$$

where

$$C_n = \frac{1}{n^2 S(12n)}. \tag{48}$$

The series (47) converges, together with all the series obtained by differentiation, uniformly in $[-\frac{1}{2}, +\frac{1}{2}]$, and $\varphi(x)$ belongs, in this interval, to $C_{\{M_n\}}$.

We have indeed by (47), and (44) in $[-\frac{1}{2}, +\frac{1}{2}]$,

$$|\varphi^{(p)}(x)| \leqq \sum_{p \leqq 3n} C_n T_{3n}{}^{(p)}(x)| \leqq \sum_{p \leqq 3n} |C_n| (12n)^p,$$

since by (44) in $[-\frac{1}{2}, \frac{1}{2}]$, $|T_{3n}{}^{(p)}(x)| \leqq \dfrac{3^p(3n)^p}{(\frac{3}{4})^p} = (12n)^p.$

And, since $p \leqq 3n < 12n$, we have, by definition of the function $S(r)$,

$$S(12n) \geqq \frac{(12n)^p}{M_p}.$$

Therefore, by (48),

$$|\varphi^{(p)}(x)| \leqq \sum_{p \leqq 3n} M_p \frac{(12n)^p}{n^2(12n)^p} \leqq M_p \sum_1^\infty \frac{1}{n^2} = CM_p.$$

This proves our assertion.

But if we suppose that $C_{\{M_n\}}$ is analytic, $\varphi(x)$, belonging to this class, is analytic in $[-\frac{1}{2}, +\frac{1}{2}]$. Let us put $x = \cos\theta$. The function $\Phi(\theta) = \varphi(\cos\theta)$ is then analytic in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, and the function $f(\psi) = \Phi\left(\frac{\psi}{3}\right)$ is analytic in the interval $[\pi, 2\pi]$. We have

$$f(\psi) = \Phi\left(\frac{\psi}{3}\right) = \varphi\left(\cos\frac{\psi}{3}\right) = \sum_1^\infty C_n T_{3n}\left(\cos\frac{\psi}{3}\right)$$

$$= \sum_1^\infty C_n \cos 3n\left(\text{arc}\cos\left(\cos\frac{\psi}{3}\right)\right) = \sum_1^\infty C_n \cos n\psi.$$

The function $f(\psi)$ being even, is also analytic in $[-\pi, \pi]$. Therefore, by Theorem III, there exists a positive constant γ such that for large n

$$|C_n| < e^{-\gamma n}.$$

That is to say, by (44),

$$\frac{1}{n^2 S(12n)} < e^{-\gamma n}.$$

And, therefore, there exists a positive constant δ, such that for large n

$$S(12n) > e^{\delta n}.$$

$S(r)$ being an increasing function, if $12n \leqq r < 12(n+1)$,

$$S(r) \geqq S(12n) > e^{\delta n} > e^{\delta(r/12-1)} > e^{\mu r},$$

where $\mu > 0$. Therefore we have proved that $S(r) > e^{\mu r}$, if $C_{\{M_n\}}$ is an analytic class.

Suppose now that the condition of the theorem is fulfilled, and let $f(x)$ belong in $[-1, +1]$ to $C_{\{M_n\}}$.

From Lemma VI it results that there exists a constant $A > 0$ such that

$$f(x) = \sum a_n T_n(x),$$

with

$$|a_n| < \frac{1}{S\left(\frac{n}{A}\right)}.$$

Therefore if n is sufficiently large, $n \geqq n_0$,

$$|a_n| < e^{-\frac{\mu}{A}n}.$$

If z is in the closed ellipse C_R with $R = e^{\delta}$ we have, by (42),

$$|T_n(z)| < e^{n\delta},$$

and, if we choose $\delta < \frac{\mu}{A}$, we have for $n \geqq n_0$

$$|a_n T_n(z)| < e^{\left(\delta - \frac{\mu}{A}\right)n} = e^{\nu n},$$

with the constant $\nu < 0$. Thus

$$\sum_{n=n_0}^{\infty} |a_n T_n(z)| < \sum_{n_0}^{\infty} e^{\nu n}$$

converges uniformly in C_R. And $\sum_1^{\infty} a_n T_n(z)$ is analytic in C_R, and particularly in $[-1, +1]$. Our theorem is thus completely proved.

The fundamental Theorem II follows immediately from the Theorem IV and from the Lemma III[11], since by this Lemma the two conditions $S(r) > e^{\mu r}$, for large r, and $(M_n^{o})^{1/n} = O(n)$, are equivalent.

8. EXAMPLES OF CLASSES WHICH ARE NOT ANALYTIC

It follows from Theorem II, that if

$$\lim_{n=\infty} \frac{M_n^{1/n}}{n} = \infty,$$

the class $C_{\{M_n\}}$ is certainly not analytic. Indeed, for the principal sequence of indices (of the sequence $\{\log M_n\}$) corresponding to the exponential regularization, $M^o_{n_i} = M_{n_i}$. Therefore

$$\overline{\lim_{n=\infty}} \frac{(M^o_n)^{1/n}}{n} \geqq \lim_{i=\infty} \frac{(M^o_{n_i})^{1/n_i}}{n_i} = \infty,$$

and the condition for analyticity is not fulfilled for the class $C_{\{M_n\}}$ (Theorem II).

Thus the class $C_{\{\Gamma(\alpha n)\}}$ with $\alpha > 1$ is not analytic, and in particular the classes of Gevray are not analytic.[1]

It will be useful to give the following example of a non-analytic class. As we shall see later, this class has many interesting properties.

Consider three sequences of strictly increasing positive integers $\{k_i\}$, $\{\lambda_i\}$, $\{n_i\}$ having the following properties. For $i \geqq 1$,

$$k_i > \lambda_i > 1, \quad n_{i+1} = k_i n_i, \quad n_1 = 1,$$

$$\lim \frac{\lambda_i}{\log n_i} = \lim \frac{k_i}{\lambda_i} = \lim \frac{\lambda_i}{\log k_i} = \infty.$$

This occurs, for instance, if $\{\lambda_i\}$ is any increasing sequence of integers with

$$\lambda_1 \geqq e^2, \ \lim \frac{\lambda_i}{i \log \lambda_i} \to \infty,$$

the other two sequences being defined by

$$k_i = [\lambda_i \log \lambda_i], \ n_1 = 1, \ n_{i+1} = k_i n_i \quad (i \geqq 1).$$

Let now the quantities β_{n_i} $(i \geqq 1)$ satisfy the conditions

$$0 < \beta_1 < \infty, \ 0 \leqq \beta_{n_{i+1}} \leqq \beta_{n_i} + (n_{i+1} - n_i) \log n_{i+1} \quad (i \geqq 1).$$

[1]The classes $C\{\Gamma(\alpha n)\}$ and $C\{\Gamma(\alpha n+1)\}$ are obviously the same.

Through each point (n_i, β_{n_i}) draw the straight line of slope $\log n_{i+1}$, and, for $n_i \leqq n < n_{i+1}$, denote by β_n the ordinate of the point on this line having the abscissa n. That is to say,

$$\beta_n = \beta_{n_i} + (n - n_i)\log n_{i+1}.$$

Denote now by $\{\alpha_n\}$ any sequence such that

$$\alpha_{n_i} = \beta_{n_i} \quad (i \geqq 1), \qquad \alpha_n \geqq \beta_n \quad (n \geqq 1).$$

It is clear that the segment of the straight line of slope $\log n_i$ passing through (n_i, β_{n_i}) and admitting as projection on Ox the segment $n_i \leqq x < n_{i+1}$ is the segment which, in regularizing exponentially $\{\alpha_n\}$, we have to call L_i, and the set of these segments constitutes the *exp*-base of $\{\alpha_n\}$ (the sequence $\{n_i\}$ is here the principal sequence of indices), therefore $\alpha_n{}^{(\text{exp})} = \beta_n$. If $n_i \leqq n < n_{i+1}$, we have

$$\frac{\alpha_n^{(\text{exp})}}{n - n_i} = \frac{\beta_n}{n - n_i} \geqq \frac{\beta_n - \beta_{n_i}}{n - n_i} = \log n_{i+1} = \log k_i + \log n_i.$$

Thus

$$\frac{\alpha_n^{(\text{exp})}}{n} - \log n \geqq \left(1 - \frac{n_i}{n}\right) (\log k_i + \log n_i) - \log n.$$

If we choose

$$n = \lambda_i n_i < k_i n_i = n_{i+1},$$

we get, for these values of n,

$$\frac{\alpha_n^{(\text{exp})}}{n} - \log n \geqq \left(1 - \frac{1}{\lambda_i}\right) (\log k_i + \log n_i) - \log n_i - \log \lambda_i$$

$$= \log \frac{k_i}{\lambda_i} - \frac{\log k_i}{\lambda_i} - \frac{\log n_i}{\lambda_i},$$

and by the properties of the sequences,

$$\lim_{\substack{n = \lambda_i n_i \\ i = \infty}} \left(\frac{\alpha_n^{(\text{exp})}}{n} - \log n\right) = \infty. \tag{49}$$

If we put $M_n = e^{\alpha_n}$, we have $M_n^o = e^{\alpha_n^{(\text{exp})}}$, and by (47),

$$\overline{\lim_{n=\infty}} \frac{(M_n^o)^{1/n}}{n} = \infty.$$

Thus the class $C_{\{M_n\}}$ is not analytic.

In choosing in different manners the sequence $\{\beta_{ni}\}$, we may construct non-analytic classes having various interesting properties.

9. CLASSES OF PERIODIC FUNCTIONS

We have seen that, in order to prove theorems concerning general i. d. functions, it is convenient to proceed by way of the study of periodic i. d. functions. It is, as a matter of fact, easier, from our point of view, to study Fourier series and, as we shall see, conditions for analyticity of classes composed only of periodic functions, or even conditions for equivalence of two classes composed only of periodic functions, are much simpler than those concerning general i. d. classes. The chief reason for this is that in the periodic case we have to use convex regularization, instead of the exponential one which must be used in the general case.

We shall denote by $C^*_{\{M_n\}}$ the subclass of $C_{\{M_n\}}$ defined in $[0, 2\pi]$, composed of all the functions periodic, together with their derivatives, with the period 2π. That is to say, $C^*_{\{M_n\}}$ is the class of all i. d. functions defined for all values of x, periodic with period 2π, and such that to every function $f(x)$ of this class corresponds a constant $k=k(f)$ such that for every x,

$$|f^{(n)}(x)| < k^n M_n (n \geqq 1). \tag{50}$$

We can also say that $C^*_{\{M_n\}}$ is the class of all i. d. functions $f(x)$ in $[0, 2\pi]$ such that $f^{(n)}(0) = f^{(n)}(2\pi)$ $(n \geqq 1)$ and such that (50) holds.

Every function of a class $C^*_{\{M_n\}}$ can be expanded in a Fourier series

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{51}$$

uniformly convergent in $[0, 2\pi]$. $f^{(n)}(x)$ is given by the series obtained by differentiating n times (51). Each of these series converges to $f^{(n)}(x)$ uniformly.

A class $C^*_{\{M_n\}}$ will be called a trigonometric class, or the trigonometric subclass of the class $C_{\{M_n\}}$.

The following Lemma is important for the study of trigonometric classes.

LEMMA VII. *If*

$$f(x) = \sum (a_n \cos nx + b_n \sin nx)$$

belongs to $C^*_{\{M_n\}}$, *there exists a constant* $A > 0$ *such that for* $n \geqq 1$,

$$\left.\begin{aligned} |a_n| &< \frac{1}{T\left(\dfrac{n}{A}\right)}, \\ |b_n| &< \frac{1}{T\left(\dfrac{n}{A}\right)}, \end{aligned}\right\} \qquad (n \geqq 1)$$

where

$$T(r) = \overline{\operatorname{Bd}}_{n \geqq 1} \frac{r^n}{M_n}.$$

It is immediately seen that, if

$$\varliminf (M_n)^{1/n} = a < \infty, \tag{52}$$

then, for $r > a$: $T(r) = \infty$, since to every $\epsilon > 0$ corresponds a sequence $\{n_j\}$ such that $M_{n_j} < (a+\epsilon)^{n_j}$ $(j \geqq 1)$, and therefore if $r > a + \epsilon$,

$$T(r) \geqq \overline{\operatorname{Bd}}_{j \geqq 1} \left(\frac{r}{a+\epsilon}\right)^{n_j} = \infty.$$

If, in particular,

$$\varliminf (M_n)^{1/n} = 0, \tag{53}$$

then

$$T(r) = \infty \; (r > 0).$$

Therefore the theorem we are considering shows that if (52) holds, $f(x)$ is a trigonometric polynomial, i. e.,

$$f(x) = \sum_0^m (a_n \cos nx + b_n \sin nx).$$

If (53) holds, $f(x)$ is necessarily a constant.

If now

$$\lim M_n^{1/n} = \infty,$$

the sequence $\{M_n\}$ can be regularized convexly by means of logarithms, and $T(r)$ can be written in the form

$$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n}.$$

As we have seen, in Theorem III, for $n \geqq 1$,

$$a_n = \frac{\mp 1}{n^p \pi} \int_0^{2\pi} f^{(p)}(x) \begin{matrix} \sin nx \\ \text{or} \\ \cos nx \end{matrix} dx, \ (p \geqq 0).$$

Since $f(x)$ belongs to $C^*_{\{M_n\}}$, we have

$$|a_n| \leqq 2\frac{k^p M_p}{n^p} < \left(\frac{A}{n}\right)^p M_p (p \geqq 1).$$

And therefore

$$|a_n| \leqq \frac{1}{\overline{\operatorname*{Bd}}_{p \geqq 1} \dfrac{\left(\dfrac{n}{A}\right)^p}{M_p}} = \frac{1}{T\left(\dfrac{n}{A}\right)}.$$

The same inequality holds for $|b_n|$, and the Lemma is proved.

We are now able to prove a theorem analogous to Theorem IV, this time concerning trigonometric classes.

THEOREM V. *A necessary and sufficient condition for a trigonometric class $C^*_{\{M_n\}}$ to be analytic is that there exist a positive constant μ such that, for large values of r,*

$$T(r) > e^{\mu r}, \tag{54}$$

where

$$T(r) = \overline{\operatorname*{Bd}}_{n \geqq 1} \frac{r^n}{M_n}.$$

We have seen, page 54, that if $\underline{\lim}\, M_n^{1/n} < \infty$, $T(r) = \infty$ for large values of r, and the only functions belonging to $C^*_{\{M_n\}}$ are trigonometric polynomials. The condition (54) is then satisfied, and evidently the class is analytic. We have then to prove our theorem when $\lim M_n^{1/n} = \infty$, i. e.,

$$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n}.$$

If $f(x)$ belongs to $C^*\{M_n\}$, by Lemma VII, there exists a constant $A>0$, such that in writing

$$f(x) = \sum (a_n \cos nx + b_n \sin nx),$$

we have

$$\left.\begin{aligned} |a_n| &< \frac{1}{T\left(\dfrac{n}{A}\right)}, \\ |b_n| &< \frac{1}{T\left(\dfrac{n}{A}\right)}. \end{aligned}\right\} \quad (n \geqq 1)$$

If (54) is satisfied, we get

$$|a_n| < e^{-\frac{\mu n}{A}},$$
$$|b_n| < e^{-\frac{\mu n}{A}},$$

and by Theorem III, $f(x)$ is analytic.

In order to prove the necessity of the condition (54), consider the function

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 T(n)}.$$

This function belongs to $C^*\{M_n\}$. Indeed

$$\varphi^{(p)}(x) = \pm \sum \frac{n^p}{n^2 T(n)} \begin{Bmatrix} \cos nx \\ \text{or} \\ \sin nx \end{Bmatrix},$$

therefore

$$|\varphi^{(p)}(x)| \leqq \sum \frac{n^p}{n^2 T(n)},$$

but, since

$$T(n) \geqq \frac{n^p}{M_p} \ (p \geqq 1),$$

$$|\varphi^{(p)}(x)| \leqq M_p \sum \frac{1}{n^2} = C M_p \ (p \geqq 1),$$

and $\varphi(x) \in C^*\{M_n\}$.

If $C^*_{\{M_n\}}$ is analytic, $\varphi(x)$, belonging to this class, is analytic, and, by Theorem III, there exists a positive constant γ such that, for large values of n,

$$\frac{1}{n^2 T(n)} < e^{-\gamma n}.$$

If $n \leqq r < n+1$, we have, for large values of r,

$$T(r) > T(n) > \frac{1}{n^2} e^{\gamma n} > e^{\alpha n} > e^{\mu r},$$

and the theorem is proved.

From this theorem, and Lemma III(IV), and from the remarks on page 54, there follows the following fundamental theorem for trigonometric classes.

THEOREM VI. *If*

$$\underline{\lim}\, M_n^{1/n} < \infty,$$

the trigonometric class $C^*_{\{M_n\}}$ *is composed exclusively of trigonometric polynomials. If*

$$\underline{\lim}\, M_n^{1/n} = 0,$$

$C^*_{\{M_n\}}$ *contains constants only. If*

$$\lim M_n^{1/n} = \infty,$$

a necessary and sufficient condition that $C^*_{\{M_n\}}$ *be analytic is that*

$$(M_n^c)^{1/n} = O(n),$$

where $\{M_n^c\}$ *is the convex regularized sequence of the sequence* $\{M_n\}$ *regularized by means of logarithms.*

If we return to the examples constructed on page 51, we see that if $\beta_{n_i} = 0 (i \geqq 1)$, then $M_{n_i} = M_{n_i}^c = 1$, and the class $C^*_{\{M_n\}}$ is composed only of trigonometric polynomials. That is to say, the class $C_{\{M_n\}}$ is not analytic (contains functions which are not analytic) but all the functions which are periodic together with their derivatives (with the period 2π, for instance) are not only analytic, but each of these functions is a sum of a finite number of harmonics.

But it is also possible to construct a sequence $\{M_n\}$ with $\lim M_n^{1/n}=\infty$, and yet such that the class $C_{\{M_n\}}$ is non-analytic, its trigonometric subclass $C^*_{\{M_n\}}$ being analytic.

The sequences $\{k_i\}$, $\{n_i\}$ being those of page 51, let us put

$$\delta_i=\frac{\log n_{i-1}}{i}\ (i\geqq 3).$$

From $n_1=1$, $n_{i+1}=k_in_i(i\geqq 1)$ it follows that

$$n_i=k_1k_2\cdots k_{i-1},$$

$$\delta_i=\frac{\log k_1+\log k_2\cdots+\log k_{i-2}}{i}.$$

Since $k_i\uparrow\infty$, we have

$$\delta_i>0,\ \delta_{i+1}>\delta_i,\ \lim\delta_i=\infty.$$

Put now

$$\beta_1=0,\ \beta_{n_2}=0,\ \beta_{n_i}=\delta_3(n_3-n_2)+\cdots+\delta_i(n_i-n_{i-1})\ (i\geqq 3).$$

Thus

$$\frac{\beta_{n_{i+1}}-\beta_{n_i}}{n_{i+1}-n_i}=\delta_{i+1}=\frac{\log n_i}{i+1}<\log n_i,$$

and

$$\beta_{n_i}\leqq i\delta_in_i\leqq n_i\log n_{i-1}<n_i\log n_i.$$

It follows from the last two inequalities that, putting $P_n\equiv(n, \log(n!)+pn)$, if p is large, all the points (n_i, β_{n_i}) are below the corresponding points $P_{n_i}(i\geqq 1)$, and the slope of the segment joining P_{n_i} to P_{n_i+1} is greater than the slope of the corresponding segment joining (n_i, β_{n_i}) to $(n_{i+1}, \beta_{n_{i+1}})$. If we construct now the sequence $\{\alpha_n\}$, as on page 52, we see immediately that

$$\alpha_n^{(\infty)}\leqq\log n!+pn,\ (n>n_3),$$

and

$$M_n^c<n!e^{pn}.$$

That is to say,

$$(M_n^c)^{1/n}=O(n),$$

and the class $C^*_{\{M_n\}}$ is analytic.

We see on the other hand that, if $n_i < n < n_{i+1} (i > 2)$,

$$\alpha_n^{(\infty)} = \delta_3(n_3 - n_2) + \cdots + \delta_i(n_i - n_{i-1}) + \delta_{i+1}(n - n_i),$$

and therefore if $3 < k < i+1$,

$$\frac{\alpha_n^{(\infty)}}{n} > \frac{\delta_k(n - n_k)}{n}.$$

That is to say,

$$\underline{\lim}\ \frac{\alpha_n^{(\infty)}}{n} \geqq \delta_k;$$

and since $\delta_k \to \infty$,

$$\lim \frac{\alpha_n}{n} = \lim \frac{\alpha_n^{(\infty)}}{n} = \infty\ ;$$

which shows that

$$\lim\ (M_n)^{1/n} = \infty.$$

The properties of our example are then proved.

10. DIFFERENTIABLE CLASSES

A class $C_{\{M_n\}}$, or $C^*_{\{M_n\}}$, will be said to be differentiable, if from the fact that $f(x)$ belongs to $C_{\{M_n\}}$, respectively to $C^*_{\{M_n\}}$, it follows that $f'(x)$ belongs to $C_{\{M_n\}}$, respectively to $C^*_{\{M_n\}}$. It is clear that if $f(x)$ belongs to a differentiable class, then all the derivatives $f^{(n)}(x)$ belong to this class.

In this paragraph we shall give conditions for the differentiability of a trigonometric class. Such conditions for a class $C_{\{M_n\}}$ will be given later.

It is obvious that if $\underline{\lim}\ M_n^{1/n} = M < \infty$, the class $C^*_{\{M_n\}}$ is differentiable, since, if $M = 0$, this class contains only constants (see page 55), and if $M > 0$, this class is simply the class of all trigonometric polynomials. For the more important case which remains, namely, $\lim\ M_n^{1/n} = \infty$, we shall prove the following theorem.

THEOREM VII. *If* $\lim\ M_n^{1/n} = \infty$, *a necessary and sufficient condition in order that* $C^*_{\{M_n\}}$ *be differentiable, is that*

(55) $$\varliminf_{r=\infty} \frac{\log T(r)}{(\log r)^2} > 0,$$

where $$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n}.$$

Suppose that (55) is satisfied, and let $f(x) = \sum (a_n \cos nx + b_n \sin nx)$ be a function of $C^*_{\{Mn\}}$. By Lemma VII,

$$|a_n| < \frac{1}{T(\alpha n)}, \ |b_n| < \frac{1}{T(\alpha n)} \ (n \geqq 1),$$

where α is a positive constant. Thus

$$|f^{(p+1)}(x)| \leqq \sum (|a_n| + |b_n|) n^{p+1} < 2 \sum \frac{n^{p+1}}{T(\alpha n)}.$$

For any $\beta > 0$, since $T(\beta n) \geqq \frac{\beta^p n^p}{M_p}$, $(p \geqq 1)$, we may write

(56) $$|f^{(p+1)}(x)| \leqq 2 \sum \frac{n^{p+1}}{T(\beta n)} \cdot \frac{T(\beta n)}{T(\alpha n)} \leqq \frac{2M_p}{\beta^p} \sum n \frac{T(\beta n)}{T(\alpha n)},$$

the last series being convergent if β is sufficiently small.

Indeed, by formula (38), if $\beta < \alpha$,

(57) $$n \frac{T(\beta n)}{T(\alpha n)} = e^{\log n - \int_{\beta n}^{\alpha n} \frac{N(t)}{t} dt} \leqq e^{\log n - N(\beta n) \log\left(\frac{\alpha}{\beta}\right)}$$

and, on the other hand, if we put

$$\varliminf_{r=\infty} \frac{\log T(r)}{(\log r)^2} = A,$$

it follows from Lemma V (1), that

$$\varliminf_{r=\infty} \frac{N(r)}{\log r} \geqq A,$$

in other words, for $r \geqq 1$ (since $N(r)$ is greater or equal to unity), $N(r) > B \log r$, where B is a positive constant. Thus (57) gives

$$n\frac{T(\beta n)}{T(\alpha n)} < e^{-\log n - B\log(\beta n)\log\left(\frac{\alpha}{\beta}\right)} = e^{\left(1 - B\log\left(\frac{\alpha}{\beta}\right)\right)\log n - B\log\beta .\log\left(\frac{\beta}{\alpha}\right)},$$

and if β is chosen in such a manner that

$$B\log\frac{\alpha}{\beta} > 3,$$

we have

$$n\frac{T(\beta n)}{T(\alpha n)} \leqq e^{-2\log n + C} = \frac{k}{n^2},$$

and the last series in (56) converges. Therefore

$$|f^{(p+1)}(x)| < L\beta^{-p}M_p \quad (p \geqq 1),$$

where L and β are constants. Thus, since $f^{(p+1)}(x)$ is the p-th derivative of $f'(x)$, the latter function belongs to $C^*_{\{M_n\}}$, and therefore this class is differentiable.

Suppose now that $C^*_{\{M_n\}}$ is differentiable and let us prove that (55) holds.

Let us recall in the first place, that the function

$$\varphi(x) = \sum\frac{\cos nx}{n^2 T(n)}$$

belongs to $C^*_{\{M_n\}}$ (see page 56). Therefore, if, as we suppose, the class $C^*_{\{M_n\}}$ is differentiable, then, since $\varphi(x)$ belongs to the class, $\varphi^{(3)}(x)$ must also belong to it. But

$$\varphi^{(3)}(x) = \sum\frac{n\sin nx}{T(n)},$$

and, by Lemma VII, there exists a constant A, greater than unity, we may suppose, such that

$$\frac{n}{T(n)} < \frac{1}{T\left(\frac{n}{A}\right)},$$

that is,

$$n < \frac{T(n)}{T\left(\frac{n}{A}\right)} = e^{\int_{\frac{n}{A}}^{n}\frac{N(t)}{t}dt} < e^{N(n)\log A}.$$

Thus $N(n)\log A > \log n (n \geqq 1)$. If now $n \leqq r < n+1$, for large values of n,

$$N(r)\log A \geqq N(n)\log A > \log n > k \log r,$$

where k is a positive constant. Thus, again by Lemma V (1),

$$\underline{\lim} \frac{\log T(r)}{(\log r)^2} \geqq \frac{1}{4} \underline{\lim} \frac{N(r)}{\log r} \geqq \frac{k}{\log A} > 0,$$

and the theorem is proved.

Remark that, since $T(r)$ is continuous, it follows that $\log T(e^t) = A(t) = A_c(t)$, and, by Lemma V, the condition (55) is equivalent to each of the conditions

$$\overline{\lim} \left(\frac{M^c_{n+1}}{M^c_n}\right)^{1/n} < \infty, \quad \overline{\lim} \frac{\log M^c_n}{n^2} < \infty, \quad \overline{\lim} \left(\frac{M^c_{n+1}}{M_n}\right)^{1/n} < \infty.$$

Therefore, by Theorem VII, we have the following:

THEOREM VIII[I]. *If* $\underline{\lim}\, M_n^{1/n} < \infty$, *the class* $C^*_{\{M_n\}}$ *is differentiable.*

If

$$\lim M_n^{1/n} = \infty,$$

a necessary and sufficient condition in order that the class $C^*_{\{M_n\}}$ *be differentiable is one of the following three equivalent conditions:*

$$(58) \quad \overline{\lim} \left(\frac{M^c_{n+1}}{M^c_n}\right)^{1/n} < \infty, \qquad (58') \quad \overline{\lim} \left(\frac{M^c_{n+1}}{M_n}\right)^{1/n} < \infty,$$

$$(59) \qquad \overline{\lim} \frac{\log M^c_n}{n^2} < \infty.$$

II. PROBLEM OF EQUIVALENCE OF TRIGONOMETRIC CLASSES

For the equivalence of two classes it is sufficient, as we have seen, to give conditions for the inclusion of one class in another. We have therefore to find conditions in order that $C^*_{\{M_n\}}$ be a subclass of $C^*_{\{M_n'\}}$.

Since, if $0 < \underline{\lim}\, M_n^{1/n} < \infty$, the corresponding class is the class of all trigonometric polynomials, and that, if $\underline{\lim}\, M_n^{1/n} = 0$, the class contains only constants, we can immediately settle first of all the following simple cases:

If $\underline{\lim}\, M_n^{1/n} = 0$, $C^*\{M_n\}$ *is a subclass of every class* $C^*\{M_n'\}$.

If $0 < \underline{\lim}\, M_n^{1/n} < \infty$, $C^*\{M_n\}$ *is a subclass of every class* $C^*\{M_n'\}$ *with* $\underline{\lim}(M_n')^{1/n} > 0$, *but of no class* $C^*\{M_n'\}$ *with* $\underline{\lim}(M_n')^{1/n} = 0$.

If $\lim M_n^{1/n} = \infty$, $C^*\{M_n\}$ *is not a subclass of any class* $C^*\{M_n'\}$ *with* $\underline{\lim}\, (M_n')^{1/n} < \infty$.

Indeed, as we have seen, page 56, $\varphi(x) = \sum_1^\infty \frac{\cos nx}{n^2 T(n)}$ belongs to $C^*\{M_n\}$ but, since it is not a trigonometric polynomial, this function does not belong to $C^*\{M_n'\}$.

Therefore, the only case of equivalence, and, of course, the most important to be considered, is the following:

$$\lim M_n^{1/n} = \lim\, (M_n')^{1/n} = \infty.$$

In this case, we have the following fundamental theorem:

THEOREM VIII. *If*

$$\lim M_n^{1/n} = \lim (M_n')^{1/n} = \infty,$$

a necessary condition that $C^*\{M_n\}$ *be a subclass of* $C^*\{M_n'\}$ *is each of the following equivalent conditions:*

$$(1) \quad (M_n^c)^{1/n} = O(M_n'^c)^{1/n},$$

(2) *there exists a positive constant* α, *such that for large values of* r

$$T_M(r) > T_{M'}(\alpha r),$$

where

$$T_M(r) = \operatorname*{Max}_{r \geqq 1} \frac{r^n}{M_n},$$

$$T_{M'}(r) = \operatorname*{Max}_{r \geqq 1} \frac{r^n}{M_n'}.$$

If $C^*\{M_n'\}$ *is differentiable, each of the two conditions* (1), (2) *is also sufficient that* $C^*\{M_n\}$ *be a subclass of* $C^*\{M_n'\}$.[1]

REMARK. It follows from this theorem that, if $C^*\{M_n\}$ is differentiable, the two classes $C^*\{M_n\}$ and $C^*\{M_n^c\}$ are equivalent.

Let us first prove that, if any increasing sequence of positive integers $\{m_i\}$ is given, we can extract from it a subsequence $\{k_j\}$ such that the series

$$\sum \frac{T_M\left(\dfrac{k_j}{e}\right)}{T_M(k_j)} \tag{60}$$

converges. We have indeed

$$\frac{T_M\left(\dfrac{m_i}{e}\right)}{T_M(m_i)} = e^{-\int_{m_i/e}^{m_i} \frac{N(t)}{t}\, dt} \leqq e^{-N(m_i/e)}.$$

and, since $\lim N(r) = \infty$, we see that such a subsequence $\{k_j\}$ exists. Thus, $\{m_i\}$ being given, $\{k_j\}$ being its subsequence such that (60) converges, consider the function

$$\psi(x) = \sum \frac{\cos k_j x}{T_M(k_j)}.$$

We have obviously

$$|\psi^{(p)}(x)| \leq \sum \frac{k_j{}^p}{T_M(k_j)} = \sum \frac{k_j{}^p}{T_M(k_j/e)} \cdot \frac{T_M(k_j/e)}{T_M(k_j)},$$

and, since

$$T_M\left(\frac{k_j}{e}\right) \geqq \left(\frac{k_j}{e}\right)^p \cdot \frac{1}{M_p},$$

we get

$$|\psi^{(p)}(x)| \leqq e^p M_p \sum \frac{T_M(k_j/e)}{T_M(k_j)} < L^p M_p,$$

and $\psi(x)$ belongs to $C^*\{M_n\}$. Therefore if, as we suppose, $C^*\{M_n\}$ is a subclass of the class $C^*\{M_n'\}$, $\psi(x)$ must belong to $C^*\{M'_n\}$, But then, by Lemma VII, there exists a positive constant A, such that

[1]The equivalent conditions $(M_n^c)^{1/n} = O(M'^c_n)^{1/n}$, $T_M(r) > T_{M'}(\alpha r)(r > r_0)$, are necessary and sufficient that $C^*\{M_n\}$ be a subclass of $C^*\{M_n'\}$ without the restriction of differentiability of $C^*\{M_n'\}$. This statement was proved in a recent paper by A. Gorny (7).

$$(61) \qquad \frac{1}{T_M(k_j)} < \frac{1}{T_{M'}(k_j/A)} \quad (j \geqq 1).$$

This involves the existence of a positive constant B such that

$$(62) \qquad T_M(n) > T_{M'}\left(\frac{n}{B}\right) \quad (n \geqq 1),$$

since, if this were not true, to every positive sequence, $\{B_i\}$, increasing to infinity, there would correspond a positive sequence of integers $\{n_i\}$, satisfying the inequality

$$T_M(n_i) < T_{M'}\left(\frac{n_i}{B_i}\right) \quad (i \geqq 1),$$

and it would be impossible to extract from this sequence $\{n_i\}$ a subsequence $\{k_j\}$ satisfying (61), contrary to the statement just proved. If we suppose now $n \leqq r < n+1$, we have, for large values of n,

$$T_M(r) \geqq T_M(n) > T_{M'}\left(\frac{n}{B}\right) \geqq T_{M'}(\alpha r),$$

which is equivalent, by Lemma III(III), to

$$(M_n^c)^{1/n} = O((M_n'^c)^{1/n}).$$

Let us now suppose that $C^*_{\{M'_n\}}$ is a differentiable class, and let

$$f(x) = \sum (a_n \cos nx + b_n \sin nx)$$

be a function of $C^*_{\{M_n\}}$, and suppose that, for large values of r, we have

$$T_M(r) > T_{M'}(\alpha r).$$

There exists a positive constant β such that

$$|a_n| < \frac{1}{T_M(\beta n)}, \ |b_n| < \frac{1}{T_M(\beta n)}.$$

Consider the function

$$F(x) = \sum \left(\frac{a_n}{n^2} \cos nx + \frac{b_n}{n^2} \sin nx\right) = \sum (p_n \cos nx + q_n \sin nx).$$

We have for large values of n

$$|p_n| < \frac{1}{n^2 T_M(\beta n)} < \frac{1}{n^2 T_{M'}(\alpha\beta n)},$$

$$|q_n| < \frac{1}{n^2 T_M(\beta n)} < \frac{1}{n^2 T_{M'}(\alpha\beta n)},$$

and by considerations analogous to those of page 56, we see that $F(x)$ belongs to $C_{\{M'_n\}}$. But $f(x) = -F^{(2)}(x)$, and, since the class $C_{\{M'_n\}}$ is differentiable, $F^{(2)}(x)$ and $f(x)$ belong also to $C_{\{M'_n\}}$, and the theorem is proved.

12. PROBLEM OF EQUIVALENCE AND DIFFERENTIABILITY OF GENERAL CLASSES IN AN OPEN INTERVAL

Let (a, b) be an open interval. The class $C^0_{\{M_n\}}$ defined in this interval is the set of all i. d. functions defined in (a, b), and belonging to $C_{\{M_n\}}$ in every closed interval $[\alpha, \beta]$, interior to (a, b), i.e., $a < \alpha < \beta < b$. In other words, to every function $f(x)$ belonging to $C^0_{\{M_n\}}$ in (a, b), and to every $[\alpha, \beta] \subset (a, b)$, there corresponds a positive constant $k = k(f, [\alpha, \beta])$ such that in $[\alpha, \beta]$ we have

$$|f^{(n)}(x)| < k^n M_n (n \geqq 1).$$

It is seen, by the Borel-Lebesgue theorem, that this definition is equivalent to the following one: $f(x)$ is a function of $C^0_{\{M_n\}}$ in (a, b), if to a neighborhood sufficiently small of every point x_0 of (a, b) there corresponds a constant $k > 0$ such that the preceding inequality holds for $n \geqq 1$. The classes $C^0_{\{M_n\}}$ were first introduced by H. Cartan (**3**).

The class $C^0_{\{M_n\}}$ is differentiable if from $f(x) \in C^0_{\{M_n\}}$, in (a, b), it follows that $f'(x) \in C^0_{\{M_n\}}$ in (a, b). $C^0_{\{M_n\}}$ is a subclass of $C^0_{\{M_n'\}}$ if, from $f(x) \in C^0_{\{M_n\}}$, in (a, b), it follows that $f(x) \in C^0_{\{M_n'\}}$. If $C^0_{\{M_n\}}$ is a subclass of $C^0_{\{M_n'\}}$, and conversely, the two classes are equivalent.

By considerations analogous to those involved previously, and largely based, as in the study of the analyticity of a class, on the principle of regularization (exponential regularization), and, as in the case of trigonometric series, on the introduction of lacunary series, H. Cartan and the present author gave, in a joint paper, necessary and sufficient conditions for the equivalence of two classes $C^0\{M_n\}$, $C^0\{M_n'\}$. But the complete proof of the sufficiency of these conditions requires, in addition to the exponential regularization, an inequality which gives a relation between the maxima of the derivatives of a function, which was proved, independently, by Gorny (7) and H. Cartan (3), and was based on theorems of Markoff and S. Bernstein relative to the theory of best approximation. It is, however, possible to prove the theorem on equivalence without using further new notions, provided that we make, as in the case of trigonometric classes, the restriction that the classes are differentiable (this restriction being made only for the proof of the sufficiency).

We shall then prove the following three theorems, of which the last is of fundamental importance.

THEOREM IX. *A necessary and sufficient condition that $C^0\{M_n\}$ be differentiable is each of the following two equivalent conditions*

$$\overline{\lim}\left(\frac{M^o_{n+1}}{M_n}\right)^{1/n} < \infty, \quad \overline{\lim}\left(\frac{M^o_{n+1}}{M^o_n}\right)^{1/n} < \infty. \tag{63}$$

THEOREM X. *If $C^0\{M_n\}$ is differentiable, the classes $C^0\{M_n\}$ and $C^0\{M^o_n\}$ are equivalent.*

THEOREM XI. *A necessary condition that $C^0\{M_n'\}$ be a subclass of $C^0\{M_n\}$ is that*

$$(M^o_n)^{1/n} = O(M_n')^{1/n}.[1] \tag{64}$$

[1]The three theorems explain the notation $\{M^o_n\}$ for the exponentially regularized sequence of $\{M_n\}$ regularized by means of logarithms. The index o in M^o_n comes from the word open. For the general problem of equivalence in a closed interval ($C\{M_n\}$) the regularization used is somewhat different, although closely related to the exponential one (4).

If $C^{0}\{M_n\}$ is differentiable, this condition is also sufficient in order that $C^{0}\{M_n\}$ be a subclass of $C^{0}\{M_n'\}$.[1]

We shall prove the three theorems simultaneously, but let us first prove the first part of Theorem XI. For this purpose we need some inequalities relative to Tchebytcheff polynomials. Let us put

$$Z_n(x) = \tfrac{1}{2}(T_{n-1}(x) + T_n(x)).$$

We have then the following inequality, for $p \leqq n$,

$$\left(\frac{n}{D}\right)^p \leqq |Z_n^{(p)}(0)| \leqq n^p, \tag{65}$$

where D is a positive constant.

We have indeed, by (43)

$$T_n^{(q+2)}(0) = (q^2 - n^2) T_n^{(q)}(0), \tag{66}$$

from which it follows that

$$|T_n^{(2q)}(0)| = [n^2 - (2q-2)^2]\,[n^2 - (2q-4)^2] \cdots n^2 |T_n(0)|,$$
$$|T_n^{(2q+1)}(0)| = [n^2 - (2q-1)^2] \cdots (n^2 - 1) |T_n'(0)|.$$

On the other hand, it can be seen immediately that, when $n > 1$ is even, $|T_n(0)| = 1$, $T_n'(0) = 0$, and, when $n > 1$ is odd, $T_n(0) = 0$, $|T_n'(0)| = n$.

We have therefore, when n is even,

$$|Z_n^{(2q)}(0)| = \tfrac{1}{2}|T_n^{(2q)}(0)| = \tfrac{1}{2}n^2 \cdots [n^2 - (2q-2)^2],$$
$$|Z_n^{(2q+1)}(0)| = \tfrac{1}{2}|T_{n-1}^{(2q+1)}(0)| = \frac{(n-1)}{2}[(n-1)^2 - 1] \cdots$$
$$[(n-1)^2 - (2q-1)^2],$$

and, when n is odd,

$$|Z_n^{(2q)}(0)| = \tfrac{1}{2}|T_{n-1}^{(2q)}(0)| = \tfrac{1}{2}(n-1)^2 \cdots [(n-1)^2 - (2q-2)^2],$$
$$|Z_n^{(2q+1)}(0)| = \tfrac{1}{2}|T_n^{(2q+1)}(0)| = \frac{n}{2}(n^2 - 1) \cdots [n^2 - (2q-1)^2].$$

[1]We recall that H. Cartan and Mandelbrojt proved that Theorem X is true without supposing that $C^0\{M_n\}$ is differentiable, and that (64) is sufficient in order that $C^0\{M_n\}$ be a subclass of $C^0\{M'_n\}$ without supposing that $C^0\{M_n\}$ is differentiable (4).

Thus, if we take the first of these four equalities, we have, putting $n=2r\geqq 2q$,

$$|Z_n^{(2q)}(0)|=\tfrac{1}{2}\cdot 2^{2q}[r-(q-1)]\cdots r\cdot r\cdot(r+1)\cdots[r+(q-1)]$$
$$>\frac{2^{2q}r^q r^q}{C^q}=\left(\frac{n}{C_1}\right)^{2q},$$

where C_1 is a numerical constant. We can prove an analogous inequality for the other three cases. The second part of the inequality (65) is obvious, by the four equalities. Thus (65) is proved.

It follows also, from (44), that, in every interval $[-d\ +d]$, $0<d<1$, the following inequality is satisfied

$$|Z_n^{(p)}(x)|\leqq K_d^p n^p, \tag{67}$$

where K_d is a constant depending on d.

A positive sequence of integers $\{n_i\}$ being given, we can extract from it a subsequence $\{k_j\}$ having the following properties. D being the constant involved in (65), k_1 is a term of $\{n_i\}$, such that $k_1>D$ and k_{j+1} will be defined, starting from k_j, by the inequalities

(1) $$k_{j+1}>k_j,\quad \sum_{j=1}^{\infty}\frac{1}{k_j}<\infty,$$

(2) for $1\leqq q\leqq k_j$, we shall have

$$\frac{k_{j+1}^{q+1}}{S_M(k_{j+1})}\leqq M_q,$$

(3) for every j we shall have

$$\sum_{m=j+1}^{\infty}\frac{k_m^{k_j}}{S_M(k_m)}<\frac{k_j}{2DS_M(k_j)},$$

(4) for every $1\leqq q\leqq k_j$, we shall have

$$\frac{k_{j+1}^{q}}{S_M(k_{j+1})}<D^q M_q'.$$

All these conditions can be satisfied simultaneously, since for every positive integer s,

$$\lim_{r=\infty} \frac{r^s}{S_M(r)} = 0.$$

Consider now the function

$$f(x) = \sum_{j=1}^{\infty} \frac{Z_{k_j}(x)}{S_M(k_j)}. \tag{68}$$

This function belongs, in $(-1, +1)$, to $C^0\{M_n\}$ We see, indeed, by (67), that, in every interval $[-d, d]$, $0<d<1$, if $k_{l-1}<p \leqq k_l$,

$$|f^{(p)}(x)| = \left| \sum_{k_j \geqq p} \frac{Z_{k_j}^{(p)}(x)}{S_M(k_j)} \right| \leqq K_d^p \sum_{j=l}^{\infty} \frac{k_j^p}{S_M(k_j)}$$
$$= K_d^p \left(\frac{k_l^p}{S_M(k_l)} + \sum_{j=l+1}^{\infty} \frac{k_j^p}{S_M(k_j)} \right).$$

But, if $p \leqq k_l$, we have also

$$\frac{k_l^p}{S_M(k_l)} \leqq M_p^0 \leqq M_p,$$

and from (1) and (2) it follows that

$$\sum_{j=l+1}^{\infty} \frac{k_j^p}{S_M(k_j)} \leqq M_p \sum_{j=l+1}^{\infty} \frac{1}{k_j} < AM_p.$$

Thus

$$|f^{(p)}(x)| < K_d^p(1+A)M_p < R_d^p M_p,$$

and $f(x)$ belongs, in $(-1, +1)$, to $C^0\{M_n\}$. But by our hypotheses $C^0\{M_n\}$ is a subclass of $C^0\{M_n'\}$, therefore $f(x)$ belongs, in $(-1, +1)$, to $C^0\{M_n'\}$. Thus there exists a constant $L>1$, such that for $k_{l-1}<p \leqq k_l$

$$L^p M_p' > |f^{(p)}(0)| = \left| \sum_{j=l}^{\infty} \frac{Z_{k_j}^{(p)}(0)}{S_M(k_j)} \right| \geqq \left| \frac{Z_{k_l}^{(p)}(0)}{S_M(k_l)} \right| - \left| \sum_{j=l+1}^{\infty} \frac{Z_{k_j}^{(p)}(0)}{S_M(k_j)} \right|.$$

By the inequalities (65) and by the property (3) of the sequence $\{k_j\}$, we have then, since $\left(\frac{k_j}{D}\right)^p \geqq \frac{k_j}{D}$,

$$L^p M_p' > \left(\frac{k_l}{D}\right)^p \frac{1}{S_M(k_l)} - \sum_{j=l+1}^{\infty} \frac{k_j^p}{S_M(k_j)} \geqq \frac{1}{2}\left(\frac{k_l}{D}\right)^p \frac{1}{S_M(k_l)}.$$

Thus, for $k_{l-1}<p\leqq k_l$, we have

$$\frac{k_l^p}{S_M(k_l)}\leqq 2(DL)^p M_p' < S^p M_p' \quad (S \text{ constant}). \tag{69}$$

By the property (4) this inequality is also true when $1\leqq p\leqq k_{l-1}$. We have thus proved that, from every positive sequence of integers $\{n_i\}$, we can extract a subsequence $\{k_j\}$, and determine a constant S, such that

$$S_M(k_j)\geqq \operatorname*{Max}_{p\leqq k_j}\frac{k_j^p}{S^p M_p'} \quad (j=1, 2,\cdots). \tag{70}$$

It follows from this remark that there exists a constant P such that, for $n\geqq 1$,

$$S_M(n)\geqq \operatorname*{Max}_{p\leqq n}\frac{n^p}{P^p M_p'}, \tag{71}$$

since, if this were not true, to every increasing to infinity sequence $\{P_i\}$ there would correspond a sequence $\{n_i\}$ such that

$$S_M(n_i)<\operatorname*{Max}_{p\leqq n_i}\frac{n_i^p}{P_i^p M_p'},$$

and it would be impossible to extract from this sequence, $\{n_i\}$, a subsequence $\{k_j\}$ satisfying (70), contrary to the proved statement. If now $n\leqq r<n+1$, we have for large values of n

$$S_M(r)\geqq S_M(n)\geqq \operatorname*{Max}_{p\leqq n}\frac{n^p}{P^p M_p'}\geqq \operatorname*{Max}_{p\leqq n}\frac{r^p}{(2P)^p M_p'}$$
$$=\operatorname*{Max}_{p\leqq r}\frac{r^p}{(2P)^p M_p'}.$$

Denoting now the expression $(2P)^p M_p'$ by N_p, we see that for large values of r

$$S_M(r)\geqq S_N(r);$$

by Lemma III (I) we get thus, for large values of n

$$M_n^o\leqq N_n^o\leqq N_n=(2P)^n M_n',$$

and therefore we have

$$\varlimsup_{n=\infty}\left(\frac{M_n^o}{M_n'}\right)^{1/n}<\infty.$$

The first part of Theorem XI is thus proved.

We shall now prove the first part of Theorem IX. If $C^o\{M_n\}$ is differentiable, the class $C^o\{M_{n-1}\}$ (i. e., the class $C^o\{N_n\}$ with N_1 arbitrary, and $N_n=M_{n-1}$, for $n\geqq 2$) is also differentiable, since, if $f\epsilon C^o\{M_{n-1}\}$, then $f'(x)\epsilon C^o\{M_n\}$, and, by the differentiability of $C^o\{M_n\}$, $f''(x)\epsilon C^o\{M_n\}$, that is to say, $f'(x)\epsilon C^o\{M_{n-1}\}$, and $C^o\{M_{n-1}\}$ is differentiable. But this proves that the class $C^o\{M_n\}$ is a subclass of $C^o\{M_{n-1}\}$, for if $f(x)\epsilon C^o\{M_n\}$, $\int^x f(x)dx$ belongs to $C^o\{M_{n-1}\}$, and, since this class is differentiable, $f(x)\epsilon C^o\{M_{n-1}\}$ and $C^o\{M_n\}$ is actually a subclass of $C^o\{M_{n-1}\}$. Applying now the proved part of Theorem XI, we have

$$\overline{\lim}\left(\frac{M_n^o}{M_{n-1}}\right)^{1/n}<\infty,$$

which is obviously equivalent to the first inequality (63).

By Lemma V, (2), the first inequality (63) is equivalent to the second one.

Let us now prove the Theorem X. Suppose that $C^o\{M_n\}$ is differentiable. Then, by what precedes, (63) is satisfied.

Supposing now that $f(x)$ belongs to $C^o\{M_n\}$, in (a, b), consider an interval $[\alpha, \beta]\subset(a, b)$, with x_0 as center. In this interval we have

$$|f^{(n)}(x)|<k^nM_n. \tag{72}$$

Let x' be any fixed point of the interval

$$I_{x_0}\equiv\left[x_0-\frac{\beta-\alpha}{4},\quad x_0+\frac{\beta-\alpha}{4}\right],$$

and γ a positive constant such that

$$\gamma>\mathrm{Max}\left(Ck,\ \frac{4}{\beta-\alpha}\right), \tag{73}$$

where C is the numerical constant involved in (46), and k the constant involved in (72). The function

(74) $$F_{x'}(x)=f\left(x'+\frac{x}{\gamma}\right)$$

satisfies, in $[-1, +1]$, the inequality

$$|F_{x'}(x)|<\left(\frac{k}{\gamma}\right)^n M_n<\frac{M_n}{C^n},$$

and we may write

$$F_{x'}(x)=\sum a_n T_n(x),$$

where, by (46),

$$|a_n|<\frac{M_q}{n^q}\ (2\leqq q\leqq n).$$

Therefore, for large values of n, we have

(75) $$|a_n|\leqq \operatorname*{Min}_{2\leqq q\leqq n}\frac{M_q}{n^q}=\operatorname*{Min}_{1\leqq q\leqq n}\frac{M_q}{n^q}=\frac{1}{S_M(n)}.$$

Consider now the function

(76) $$\Phi(x)=\sum\frac{a_n T_n(x)}{n^2}.$$

This function belongs, in $(-1, +1)$ to $C^o{}_{\{M_n^o\}}$. Indeed, from (75), it follows, that

$$\left|\frac{a_n}{n^2}\right|\leqq\frac{1}{S_M(n)n^2},$$

and, by (44), we have in $[-d, d]$, $0<d<1$,

(77)

$$|\Phi^{(p)}(x)|\leqq\sum_{n\geqq p}\left|\frac{a_n T_n{}^{(p)}(x)}{n^2}\right|\leqq L_d^p\sum_{n\geqq p}\frac{n^p}{S_N(n)n^2}\leqq L_d^p M_p^o\sum_{n=1}^{\infty}\frac{1}{n^2}<D_d^p M_p^o,$$

where D_d is a constant depending only on d. Thus $\Phi(x)$ belongs to $C^o{}_{\{M_n^o\}}$.

From (77) and (63) it follows that, in $[-d, +d]$,

(78) $$|\Phi^{(p+2)}(x)|\leqq D_d^{p+2}M_{p+2}^o<D_d^{p+2}R^p M_p^o<S_d^p M_p^o,\ (p\geqq 1),$$

where S_d is a constant depending on d (and on the sequence $\{M_n\}$).

We have, on the other hand, by (43),

$$T_n^{(p+2)}(0)=(p^2-n^2)\,T_n^{(p)}(0),\ (p\geqq 0),$$

from which it follows that

$$\Phi^{(p+2)}(0)=\sum\frac{a_nT_n^{(p+2)}(0)}{n^2}=\sum\frac{p^2-n^2}{n^2}\,a_nT_n^{(p)}(0)=p^2\Phi^{(p)}(0)\\ -F_{x'}^{(p)}(0);$$

in other words, we see that

(79) $$F_{x'}^{(p)}(0)=p^2\Phi^{(p)}(0)-\Phi^{(p+2)}(0),$$

and, by (77) and (78), we get immediately the inequality

$$|F_{x'}^{(p)}(0)|<(p^2D_d^p+S_d^p)M_p^o<A^pM_p^o,$$

where A is a constant having the same value for any choice of x' in I_{x_0} But obviously, for such a value of x'

$$F_{x'}^{(p)}(0)=\gamma^{-p}f^{(p)}(x').$$

Therefore in I_{x_0}, the following equality is satisfied:

(80) $$|f^{(p)}(x)|<(A\gamma)^pM_p^o.$$

We have thus proved that there exists about every point $x_0\in(a,b)$ an interval, in which an inequality of the type (80) is satisfied. Thus $f(x)$ belongs, in (a, b), to $C^o\{M_n^o\}$, and therefore $C^o\{M_n\}$ is a subclass of $C^o\{M_n^o\}$. On the other hand, it is obvious that $C^o\{M_n^o\}$ is a subclass of $C^o\{M_n\}$, since $M_n^o\leqq M_n$ $(n\geqq 1)$. Theorem X is thus proved.

We have based the proof of Theorem X, in supposing that $C^o\{M_n\}$ is differentiable, only on the inequality (63). Thus the two classes $C^o\{M_n\}$ and $C^o\{M_n^o\}$ are equivalent if (63) is satisfied. Therefore if (63) holds, and if $f(x)$ belongs to $C\{M_n\}$, we have, in every interval $[\alpha,\beta]\subset(a,b)$,

$$|f^{(p+1)}(x)|\leqq k^{p+1}M_{p+1}^o<k^{p+1}K^pM_p^o\leqq B^pM_p,$$

and $C^o\{M_n\}$ is differentiable. Thus the second part of Theorem IX is proved.

If now (64) is satisfied, if $f(x)$ belongs in (a, b) to $C^o\{M_n\}$

and, if this class is differentiable, we have, by Theorem X and (64), in $[\alpha, \beta]\subset(a, b)$,

$$|f^{(p)}(x)| < k^p M_p^c < k^p P^p M_p',$$

and the second part of Theorem XI is also proved.

We have thus completely proved the Theorems IX, X, and XI.

It should be remarked that, while in the problem of equivalence for trigonometric classes the convex regularized sequence plays an important rôle, in the general case (for open intervals, at least) the same rôle is played by the exponentially regularized sequence. Since the proved theorems give necessary and sufficient conditions, one cannot expect to replace these regularized sequences by sequences regularized in an essentially different manner.

We will close this paragraph by mentioning that the problem of equivalence has also been solved for classes $C_{\{M_n\}}$ and $C_{\{M_n'\}}$, in a closed interval (**4**). The principles of the proof of the general theorem relative to a closed interval are the same as those for an open one, the regularization used in the first case being closely related to exponential regularization.

13. QUASI-ANALYTIC CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

The only function analytic in the neighborhood of a point, which is zero, and of which all derivatives are zero at this point, is the function identically equal to zero. In other words, if two analytic functions are equal and all their derivatives, of the same order, are equal at a point belonging to an interval, in which they are analytic, then they are identically equal. Thus the class $C_{\{n!\}}$, defined on an interval $[a, b]$, is such that if $f_1(x)\in C_{\{n!\}}$, and $f_2(x)\in C_{\{n!\}}$, and if $f_1^{(n)}(x_0)=f_2^{(n)}(x_0)\,(n\geqq 0)$, $x_0\in[a, b]$, then $f_1(x)\equiv f_2(x)$. Or what amounts to the same thing, if $f(x) \in C_{\{n!\}}$, and if $f^{(n)}(x_0)=0$ $(n \geqq 0)$, $x_0 \in [a, b]$, then $f(x) \equiv 0$.

A class $C_{\{M_n\}}$, defined on $\mathrm{I}=[a, b]$, is said to be *quasi-analytic* if, from $f_1(x) \in C_{\{M_n\}}$, $f_2(x) \in C_{\{M_n\}}$, and from $f_1^{(n)}(x_0) = f_2^{(n)}(x_0) (n \geqq 0)$, $x_0 \in [a, b]$, it follows that $f_1(x) = f_2(x)$ for any $x \in [a, b]$. This definition is equivalent to the following one: $C_{\{M_n\}}$ is quasi-analytic on $[a, b]$, if from $f(x) \in C_{\{M_n\}}$, and from $f^{(n)}(x_0)=0 (n \geqq 0)$, $x_0 \in [a, b]$ it follows that $f(x) \equiv 0$ in $[a, b]$. The equivalence of the two definitions follows at once from the fact that the sum and the difference of two functions belonging, in $[a, b]$, to $C_{\{M_n\}}$ belong also to the same class. Quasi-analyticity constitutes one of the most important properties of the class of analytic functions.

There exist classes which are not quasi-analytic. For instance the class $C_{\{(2n)!\}}$ is not quasi-analytic. It can be seen that the function defined on $[0, 1]$ by the equalities $f(x) = e^{-1/x^2}$ if $0 < x \leqq 1$, $f(0)=0$, belongs, in $[0, 1]$, to this class. Indeed consider the circle $C_a \equiv |z-a| = a$, $0<a<1$. Since, in the closed circle C_a, the function $f(z) = e^{-1/z^2}$, for $z \neq 0$, and $f(0)=0$, is continuous, and is holomorphic inside this circle, we have by Cauchy's integral formula

$$(81) \quad |f^{(n)}(a)| = \frac{n!}{2\pi}\left|\oint_{C_a} \frac{f(z)dz}{(z-a)^{n+1}}\right| \leqq n! \frac{\operatorname*{Max}_{z \in C_a} |f(z)|}{|a|^n} = n! a^{-n} e^{-1/2a}.$$

Thus in $[0, 1]$ we have

$$|f^{(n)}(x)| < n! \operatorname*{Max}_{0<a\leqq 1} a^{-n} e^{-1/2a} = n!(2n)^n e^{-n}.$$

Therefore $f(x) \in C_{\{(2n)!\}}$ in $[0, 1]$. But, on the other hand, $f^{(n)}(0)=0 (n \geqq 0)$, and $f(x)$ is not identically zero. Thus $C_{\{(2n)!\}}$ is not quasi-analytic.

As Gevray's study of the heat equation has shown, it is important, from the physical point of view, to be able to indicate conditions under which a class $C_{\{M_n\}}$ would be quasi-analytic. For example, the fact that classes $C_{\{\Gamma(\alpha n)\}}$, with

$\alpha > 1$, are not quasi-analytic has its significance in physics. Starting from these considerations, Hadamard proposed, in 1912, the following precise problem: *To give necessary and sufficient conditions bearing on the sequence* $\{M_n\}$, in order *that* $C_{\{M_n\}}$ *be quasi-analytic.*

Denjoy (**5**) gave conditions sufficient for quasi-analyticity of a class. But the problem was solved completely by Carleman (**2**), who, generalizing Denjoy's theorem and methods, gave necessary and sufficient conditions. Ostrowski (**16**) gave a new proof of Carleman's theorem; he gave also these conditions in a slightly different form.

Carleman's condition for quasi-analyticity is the following: Let us put $\mu_n = M_n^{1/n}$, and $\mu_n^* = \operatorname{Min}_{h \geqq 0} \mu_{n+h}$. A necessary and sufficient condition that $C_{\{M_n\}}$ be quasi-analytic is that

$$\sum \frac{1}{\mu_n^*} = \infty. \tag{82}$$

Ostrowski gives the conditions in the following form: A necessary and sufficient condition in order that $C_{\{M_n\}}$ be quasi-analytic is that

$$\int_1^\infty \frac{\log T(r)}{r^2} dr = \infty, \tag{83}$$

where

$$T(r) = \overline{\operatorname{Bd}}_{n \geqq 1} \frac{r^n}{M_n}.$$

Since, as we have seen (page 54), $T(r) = \infty$ (for large values of r), if $\underline{\lim}\, M_n^{1/n} < \infty$, which means that, if this last condition is satisfied, (83) is satisfied, we may say that, if $\underline{\lim}\, M_n^{1/n} < \infty$, $C_{\{M_n\}}$ is quasi-analytic and if $\lim M_n^{1/n} = \infty$, a necessary and sufficient condition in order that $C_{\{M_n\}}$ be quasi-analytic is that (83) be satisfied, where

$$T(r) = \operatorname{Max}_{n \geqq 1} \frac{r^n}{M_n}.$$

But we shall give also another condition, which seems to us the most natural. Indeed, as we shall see afterwards, the two classes $C_{\{M_n\}}$ and $C^*_{\{M_n\}}$ (the trigonometric subclass of $C_{\{M_n\}}$) are quasi-analytic together. On the other hand, we have seen, by Theorem VIII, that a class $C^*_{\{M_n\}}$ is characterized by the sequence $\{M_n^c\}$, which is the convex regularized sequence of the sequence $\{M_n\}$, regularized by means of logarithms. As a matter of fact, the Theorem VIII shows that the two classes $C^*_{\{M_n\}}$ and $C^*_{\{M_n^c\}}$ are equivalent, at least if the class $C^*_{\{M_n^c\}}$ is differentiable (and if $\lim M_n{}^{1/n} = \infty$). (From a recent theorem of Gorny (**7**), cited in the footnote, page 64, it follows that the two classes $C^*_{\{M_n\}}$ and $C^*_{\{M_n^c\}}$ are equivalent if, only, $\lim M_n{}^{1/n} = \infty$.) It is therefore desirable to give conditions for quasi-analyticity bearing on $\{M_n^c\}$. It is actually possible to give such conditions, and we shall prove the following statement (**9**).

If $\underline{\lim}\, M_n{}^{1/n} < \infty$, $C_{\{M_n\}}$ is quasi-analytic. If $\lim M_n{}^{1/n} = \infty$, a necessary and sufficient condition in order that $C_{\{M_n\}}$ be quasi-analytic, is that

$$\sum_{n=1}^{\infty} \frac{M_n^c}{M_{n+1}^c} = \infty. \tag{84}$$

We shall actually prove Ostrowski's statement and the last one. The following theorem contains the two statements.

THEOREM XII. *If* $\underline{\lim}\, M_n{}^{1/n} < \infty$, *the class* $C_{\{M_n\}}$ *is quasi-analytic. If* $\lim M_n{}^{1/n} = \infty$, *a necessary and sufficient condition in order that* $C_{\{M_n\}}$ *be quasi-analytic is each of the two equivalent conditions*, (83), (84):

$$\int_1^{\infty} \frac{\log T(r)}{r^2}\, dr = \infty, \quad \sum_{n=1}^{\infty} \frac{M_n^c}{M_{n+1}^c} = \infty,$$

where

$$T(r) = \operatorname*{Max}_{n \geqq 1} \frac{r^n}{M_n},$$

and where $\{M_n^c\}$ *is the convex regularized sequence of the sequence* $\{M_n\}$, *regularized by means of logarithms.*

That the two conditions (83) and (84) are equivalent follows from Lemma IV[1].

The proofs given by Denjoy, Carleman, and Ostrowski are based on the theory of functions holomorphic in a half-plane, and, actually, the proof of the necessity of the conditions is even more difficult than that of the sufficiency. Here we shall give a new and simple proof for the necessity of the conditions (83), (84), based only on some considerations relative to the average values of a real function. The proof for the sufficiency of the conditions will be a combination of that of Ostrowski, and of that of de la Vallée-Poussin (**19**), both largely modified.

The statement that each of the conditions (83), (84) is necessary for the quasi-analyticity of $C_{\{M_n\}}$ means that, if they are not satisfied, then it is possible to construct, in an interval $I=[a, b]$, a function $f(x)$, such that there exists a constant k satisfying the conditions

$$|f^{(n)}(x)| < k^n M_n (n \geqq 1), \quad (a \leqq x \leqq b),$$

such that at a certain point $x_0 \epsilon [a, b]$,

$$f^{(n)}(x_0) = 0 (n \geqq 0),$$

the function $f(x)$ being not identically zero in $[a, b]$. In other words, we have to prove that such a function $f(x)$, and such a point x_0 exist, if

$$\sum \frac{M_n^c}{M_{n+1}^c} < \infty. \tag{85}$$

The interval $[a, b]$, where such a function exists, is obviously arbitrary.

Our proof will be based on the following, unpublished, considerations of H. E. Bray, relative to the repeated mean-values of a function.

Consider a positive sequence $\{\mu_n\}$, such that $\sum_1^\infty \mu_n = \mu < \infty$, and let $f(x)$ be an integrable (integrable L, i.e., Lebesgue integrable) function in an interval $[c, d]$, with $d-c>2\mu$. Let us now write

$$M_n(x)=M(\mu_1, \mu_2, \cdots \mu_n; f(x))=\frac{1}{2^n\mu_1\cdots\mu_n}\int_{-\mu_1}^{\mu_1}dt_1\int_{-\mu_2}^{\mu_2}dt_2\cdots \int_{-\mu_n}^{\mu_n}f(x+t_1+t_2+\cdots+t_n)dt_n.$$

It is obvious that $M_n(x)$ is defined in $I_n=[c+\sum_1^n\mu_i, d-\sum_1^n\mu_i]$, and its values are independent of the order in which we take the quantities $\mu_1, \mu_2, \cdots \mu_n$.

It is also obvious that

$$M(\mu_1, \mu_2\cdots\mu_n; f(x))= M(\mu_1, \mu_2, \cdots \mu_k; M(\mu_{k+1}, \cdots \mu_n; f(x))). \tag{1}$$

We have

$$M_1(x)=M(\mu_1; f(x))=\frac{1}{2\mu_1}\int_{-\mu_1}^{\mu_1}f(x+t_1)dt_1=\frac{1}{2\mu_1}\int_{x-\mu_1}^{x+\mu_1}f(t)dt,$$

therefore $M_1(x)$ has a derivative almost everywhere in $[c+\mu_1, d-\mu_1]=I_1$, which is equal to $\frac{1}{2\mu_1}(f(x+\mu_1)-f(x-\mu_1))$.

The function $M_{m-1}(x)$ is continuous for any value of $m (m\geqq 2)$ in I_{m-1}, therefore, if $m\geqq 2$, $M_m(x)=M(\mu_m; M(\mu_1, \mu_2, \cdots, \mu_{m-1}; f(x)))$ has a derivative (everywhere) in I_m, which is given by the formula

$$\frac{d}{dx}M_m(x)=\frac{d}{dx}M(\mu_1, \mu_2, \cdots, \mu_m; f(x))=\frac{1}{2\mu_1}M(\mu_2, \mu_3, \cdots \mu_m; f(x+\mu_1)) -\frac{1}{2\mu_1}M(\mu_2, \mu_3, \cdots, \mu_m; f(x-\mu_1))=\frac{1}{2\mu_1}M(\mu_2, \mu_3, \cdots, \mu_m; f(x+\mu_1)-f(x-\mu_1)). \tag{2}$$

It follows immediately from (2), that, if $n \geqq 2$,

(3) $M_n(x)$ has, in I_n, continuous derivatives of order 0, 1, $\cdots, n-1$, and that $|M_n'(x)| < \underset{x \in I_1}{\text{Max}} |M_1'(x)|$.

If $\varphi(x)$ is continuous in $[c, d]$, in each interval $[\alpha, \beta] \subset [c+\mu_i, d-\mu_i]$, the following inequalities are satisfied,

$$\underset{x \in [\alpha, \beta]}{\text{Max}} M(\mu_i; \varphi(x)) \leqq \underset{x \in [\alpha - \mu_i, \beta + \mu_i]}{\text{Max}} \varphi(x),$$

$$\underset{x \in [\alpha, \beta]}{\text{Min}} M(\mu_i; \varphi(x)) \geqq \underset{x \in [\alpha - \mu_i, \beta + \mu_i]}{\text{Min}} \varphi(x).$$

Therefore in every interval $[\alpha, \beta] \subset I_n$, we have

(4) $\underset{x \in [\alpha, \beta]}{\text{Max}} M(\mu_1, \mu_2, \cdots \mu_n; \varphi(x)) \leqq \underset{x \in [\alpha - \mu_n, \beta + \mu_n]}{\text{Max}} M(\mu_1, \mu_2, \cdots,$
$\mu_{n-1}; \varphi(x))$,

$\underset{x \in [\alpha, \beta]}{\text{Min}} M(\mu_1, \mu_2, \cdots, \mu_n; \varphi(x)) \geqq \underset{x \in [\alpha - \mu_n, \beta + \mu_n]}{\text{Min}} M(\mu_1, \mu_2, \cdots, \mu_{n-1}; \varphi(x))$.

It follows from (4) that, if we denote I^μ the interval $[c+\mu, d-\mu]$, we have

$$\underset{x \in I^\mu}{\text{Max}} M_n(x) \leqq \underset{x \in I_1}{\text{Max}} M_1(x),$$

$$\underset{x \in I^\mu}{\text{Min}} M_n(x) \geqq \underset{x \in I_1}{\text{Min}} M_1(x).$$

In other words the family $\{M_n(x)\} (n \geqq 1)$ is bounded in I^μ.

We see also by (4), in denoting by $\mu^{(m)}$ the quantity $\sum_{n=m+1}^{\infty} \mu_n$, that we have, for $n > n_0$, in every interval $[\alpha, \beta] \subset I^\mu$,

$$\underset{[\alpha, \beta]}{O} M_n(x) = \underset{x \in [\alpha, \beta]}{\text{Max}} M_n(x) - \underset{x \in [\alpha, \beta]}{\text{Min}} M_n(x)$$

$$\leqq \underset{[\alpha - \mu^{(n_0)}, \beta + \mu^{(n_0)}]}{O} M_{n_0}(x) = \underset{x \in [\alpha - \mu^{(n_0)}, \beta + \mu^{(n_0)}]}{\text{Max}} M_{n_0}(x) - \underset{x \in [\alpha - \mu^{(n_0)}, \beta + \mu^{(n_0)}]}{\text{Min}} M_{n_0}(x).$$

Since the functions $M_n(x)$ are respectively continuous in I_n, it follows from (3) that the family $\{M_n(x)\}$ is equally continuous in I^μ, (i.e., to every $\epsilon > 0$, there corresponds an $\eta > 0$ such that if $x_1 \in I^\mu$, $x_2 \in I^\mu$, and $|x_1 - x_2| < \eta$, then $|M_n(x_1) - M_n(x_2)| < \epsilon$, for every n).

From (4) it follows also, for every $x_0 \epsilon I^\mu$, that, if $n > n_0$,

$$\operatorname*{Min}_{x\epsilon[x_0-\mu(n_0),\, x_0+\mu(n_0)]} M_{n_0}(x) \leqq M_n(x_0) \leqq \operatorname*{Max}_{x\epsilon[x_0-\mu(n_0),\, x_0+\mu(n_0)]} M_{n_0}(x),$$

which proves that $M_n(x)$ tends, in I^μ, to a limit. But, since the family $\{M_n\}$ is equally continuous in this interval, the sequence $M_n(x)$ tends, in I^μ, uniformly to a continuous function, which we shall denote by $M(x) = M(\mu_1, \mu_2, \cdots, \mu_n, \cdots; f(x))$, or by $M(\{\mu_n\}; f(x))$.

From

$$\mathrm{M}(x) = \lim_{n=\infty} M_n(x) = \lim_{n=\infty} \frac{1}{2\mu_1} \int_{-\mu_1}^{\mu_1} M(\mu_2, \cdots \mu_n; f(x+t))dt,$$

which is satisfied when $x \epsilon I^\mu$, it follows that

$$M(\mu_1, \mu_2, \cdots \mu_n, \cdots; f(x)) = \frac{1}{2\mu_1} \int_{-\mu_1}^{\mu_1} M(\mu_2, \mu_3, \cdots \mu_n, \cdots; f(x+t))dt.$$

Hence, in I^μ, we have

$$(5) \quad \frac{d}{dx} M(\mu_1, \mu_2, \cdots \mu_n, \cdots; f(x)) =$$

$$\frac{M(\mu_2, \mu_3, \cdots \mu_n \cdots; f(x+\mu_1)) - M(\mu_2, \mu_3, \cdots; f(x-\mu_1))}{2\mu_1}$$

$$= M\left(\mu_2, \mu_3, \cdots \mu_n, \cdots; \frac{f(x+\mu_1) - f(x-\mu_1)}{2\mu_1}\right).$$

Therefore we have finally the following statement:

(6) In the closed interval $I^\mu = [c+\mu, d-\mu]$, the function $M(x)$ is continuous, infinitely differentiable, and $M'(x)$ is given by (5).

Let us now return to our theorem on quasi-analyticity. We suppose, $\lim M_n^{1/n} = \infty$, and that (85) is satisfied. We shall put

$$(86) \qquad \mu_1 = \frac{1}{M_1^c}, \quad \frac{M_{n-1}^c}{M_n^c} = \mu_n, \ (n \geqq 2)$$

and

$$\frac{1}{M_1^c} + \sum_{n=1}^{\infty} \frac{M_n^c}{M_{n+1}^c} = \sum_{n=1}^{\infty} \mu_n = \mu.$$

In the interval $[-3\mu, 3\mu]$ consider the function $f(x)$, defined as follows:

$$f(x)=1, \quad \text{if} \quad |x| \leqq \mu,$$
$$f(x)=0, \quad \text{if} \quad \mu<|x| \leqq 3\mu,$$

and the corresponding function $M(x)=M(\mu_1, \mu_2, \cdots, \mu_n, \cdots; f(x))$.

This function has the following properties:

(I) $M(x)$ is continuous and infinitely differentiable in $[-2\mu, 2\mu]$ by (6).

(II) $M(x)$ is not identically zero, in $[-2\mu, 2\mu]$, since

$$M(0)=\lim \frac{1}{2^n \mu_1, \mu_2, \cdots, \mu_n} \int_{-\mu_1}^{\mu_1} dt_1 \int_{-\mu_2}^{\mu_2} dt_2, \cdots \int_{-\mu_n}^{\mu_n} dt_n = 1.$$

(III) $M^{(n)}(-2\mu)=M^{(n)}(2\mu)=0, \quad (n \geqq 0)$.

Let us write, indeed,

$$f_0(x)=f(x), \; f_1(x)=\frac{f_0(x+\mu_1)-f_0(x-\mu_1)}{2\mu_1}, \cdots,$$
$$f_n(x)=\frac{f_{n-1}(x+\mu_n)-f_{n-1}(x-\mu_n)}{2\mu_n}, \cdots .$$

Obviously, in $[-2\mu, 2\mu]$, we have

(87) $$M^{(n)}(x)=M(\mu_{n+1}, \mu_{n+2}, \cdots; f_n(x)).$$

Since we have in $[-3\mu, -\mu]$, and $[\mu, 3\mu]$, $f_n(x)=0$ $(n \geqq 0)$, we see immediately that $M(\mu_{n+1}, \cdots; f_n(-2\mu))=M(\mu_{n+1}, \cdots; f_n(2\mu))=0$ $(n \geqq 0)$. And (III) follows from it at once.

(IV) $M(x)$ belongs, in $[-2\mu, 2\mu]$ to $C_{\{M_n\}}$. From the definition of $f_n(x)$, we see, since $|f(x)| \leqq 1$, in $[-3\mu, 3\mu]$, that, in $[-3\mu+\sum_1^n \mu_i, 3\mu-\sum_1^n \mu_i]$,

$$|f_n(x)| \leqq \frac{1}{\mu_1, \mu_2, \cdots, \mu_n}.$$

Therefore in $[-2\mu, 2\mu]$ we have by (87),

$$|M^{(n)}(x)| \leqq \frac{1}{\mu_1, \mu_2, \cdots, \mu_n}.$$

But, by (86), we have

$$\frac{1}{\mu_1, \mu_2, \cdots, \mu_n} = M_1^c \frac{M_2^c}{M_1^c}, \cdots, \frac{M_n^c}{M_{n-1}^c} = M_n^c.$$

Thus, we have, in $[-2\mu, 2\mu]$,

$$|M^{(n)}(x)| \leqq M_n^c \leqq M_n,$$

and $M(x)$ belongs, in this interval, to $C_{\{M_n\}}$.

We have thus proved that, if (85) is satisfied, the class $C_{\{M_n\}}$ is not quasi-analytic.

Let us now prove that (83) (or (84)) is sufficient for the quasi-analyticity of $C_{\{M_n\}}$.

We have to prove that, if there exists, in $[a, b]$, a function $f(x)$ belonging to $C_{\{M_n\}}$ which is such that, at a point $x_0 \epsilon [a, b]: f^{(n)}(x_0) = 0$ $(n \geqq 0)$, and which is not identically zero in this interval, then (85) is satisfied. We can suppose that $x_0 = a$, since if this were not true, we could replace $[a, b]$ by that of the two intervals $[x_0, b]$, or $[-x_0 - a]$ in which respectively $f(x)$ or $f(-x)$ is not identically zero, and we should have a function which belongs to $C_{\{M_n\}}$, which is not identically zero, and is itself, with all of its derivatives, equal to zero at the left-hand extremity of this interval.

In order to prove the desired statement, we shall need the following Lemma.

LEMMA VIII. *If there exists, in $[a, b]$, a function $f(x)$, not identically zero, belonging to $C_{\{M_n\}}$, and such that $f^{(n)}(a) = 0$, $(n \geqq 0)$, then there exists, in $[0, 1]$, a function $\varphi(x)$, not identically zero, such that $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$, $(n \geqq 0)$, and such that in $[0, 1]$,*

$$|\varphi^{(n)}(x)| < M_n (n \geqq 1),$$
$$\varphi(x) \geqq 0, \quad \varphi(x) = \varphi(1-x).$$

Since $f(x)$ is not identically zero, there exists a quantity c, $a \leqq c < b$, such that $f^{(n)}(c) = 0$, $(n \geqq 0)$, and such that, for no positive ϵ, $f(x)$ is identically zero in $[c, c+\epsilon]$. We have, in $[c, b]$,

$$|f^{(n)}(x)| < k^n M_n.$$

Let us put

$$f^{(-2)}(x) = \int_0^x dt \int_0^t f(\tau) d\tau.$$

If α is any quantity such that $0 < \alpha < b - c$, the function defined by the equality $f_1(x) = f^{(-2)}(\alpha x + c)$, is not identically zero in $[0, 1]$, and is such that

$$f_1^{(n)}(0) = 0, \quad (n \geqq 0),$$

$$(88) \qquad |f_1^{(n)}(x)| < \alpha^n k^{n-2} M_{n-2} = M'_{n-2} \quad (n \geqq 3).$$

Consider now the function $f_2(x) : f_2(x) = f_1(x - x^2)$. In putting $x - x^2 = y$, $y' = 1 - 2x = y_1$, we see that

$$(89) \qquad f_2(x) = f_1(y),$$

$$f_2'(x) = f_1'(y) y_1,$$

$$f_2^{(n)}(x) = f_1^{(n)}(y) y_1^n +$$

$$\sum_{k=1}^{[n/2]} (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{k!} f_1^{(n-k)}(y) y_1^{n-2k} \quad (n \geqq 2).$$

The last formula (89) can be easily proved by mathematical induction. It is true for $n = 2$; by differentiation we prove that, if (89) is true for $n = q$, it is also true for $n = q + 1$.

We have obviously $f_2^{(n)}(0) = f_2^{(n)}(1) = 0, \quad (n \geqq 0)$.
We have also, in $[0, 1]$, for $n \geqq 3$, by (89),

$$(90) \qquad |f_2^{(n)}(x)| \leqq m_n + \sum_{k=1}^{[n/2]} \frac{n(n-1)\cdots(n-2k+1)}{k!} m_{n-k},$$

where m_p is the maximum of $|f_1^{(p)}(x)|$ in $[0, 1]$.

But, on the other hand, we have

$$f_1^{(n-k)}(x) = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} f_1^{(n)}(t) \mathrm{d}t;$$

thus, by (88), for $n \geqq 3$,

$$m_{n-k} \leqq \frac{M'_{n-2}}{(k-1)!} \int_0^1 (1-t)^{k-1} dt = \frac{M'_{n-2}}{k!}.$$

We have, therefore, by (90),

$$(91)\qquad |f_2^{(n)}(x)| \leqq M'_{n-2}\left(1+\sum_{1}^{[n/2]}\frac{n(n-1)\cdots(n-2k+1)}{(k!)^2}\right)$$

$$< M'_{n-2}\sum_{1}^{\infty}\frac{n^{2k}}{(k!)^2} < M'_{n-2}\sum_{k=1}^{\infty}\frac{(2n)^{2k}}{(2k)!} < M'_{n-2}\sum_{p=1}^{\infty}\frac{(2n)^p}{p!} = e^{2n}M'_{n-2}.$$[1]

Consider, in $[-1, +1]$, the function

$$A(x) = (f_2(x))^2.$$

We have, in this interval,

$$(92)\qquad A^{(n)}(x) = \sum_{k=0}^{n} C_n^k f_2^{(k)}(x) f_2^{(n-k)}(x).$$

But the function

$$\Theta(y) = f_2\left(\frac{y+\pi}{2\pi}\right)$$

is, in $[-\pi, \pi]$, an even function, with $\Theta^{(n)}(-\pi) = \Theta^{(n)}(\pi) = 0$, thus

$$\Theta(y) = \frac{d_0}{2} + \sum_{q=1}^{\infty} d_q \cos qy,$$

with

$$d_q = \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta(y)\cos(qy)\,dy,$$

and

$$f_2(x) = \frac{d_0}{2} + \sum_{q=1}^{\infty}(-1)^q d_q \cos(2\pi q x),$$

$$f_2^{(l)}(x) = 2(\pi)^l \sum_{q=1}^{\infty} \pm d_q q^l \begin{matrix}\cos(2\pi q x)\\ \text{or}\\ \sin(2\pi q x)\end{matrix},$$

$$(93)\qquad d_q = 2\,(-1)^q \int_0^1 f_2(x)\,\cos\,(2\pi q x)\,dx.$$

Thus we have

$$|f_2^{(l)}(x)| \leqq (2\pi)^l \sum_{q=1}^{\infty} |d_q|\, q^l,$$

and, by (92), we have in $[-1, +1]$,

[1]The original proof of this inequality (with other constants) was different from that given here. The proof given here, due to Kuzmin, was communicated to the author orally.

$$(94) \qquad |A^{(n)}(x)| \leqq (2\pi)^n \Big\{ 2\,(\frac{|d_0|}{2} + \sum |d_q|\,)\,(\sum |d_q| q^n) + \sum_{k=1}^{n-1} C_n^k \,(\sum_{q=1}^{\infty} |d_q| q^k)(\sum_{q=1}^{\infty} |d_q| q^{n-k}) \Big\}.$$

If we use now Hölder's inequality

$$\sum u_q v_q \leqq (\sum u_q^{\alpha})^{1/\alpha} \cdot (\sum v_q^{\beta})^{1/\beta} \left(\frac{1}{\alpha}+\frac{1}{\beta}=1,\ u_q \geqq 0,\ v_q \geqq 0\right),$$

where we put

$$u_q = |d_q|^{k/n} q^k;\ v_q = |d_q|^{(n-k)/n};\ \alpha = \frac{n}{k};\ \beta = \frac{n}{n-k},$$

we get

$$\sum_{q=1}^{\infty} |d_q| q^k \leqq (\sum_{q=1}^{\infty} |d_q| q^n)^{k/n} \cdot (\sum_{q=1}^{\infty} |d_q|)^{(n-k)/n}.$$

We can prove in an analogous manner that

$$\sum_{q=1}^{\infty} |d_q| q^{n-k} \leqq (\sum_{q=1}^{\infty} |d_q| q^n)^{(n-k)/n} \cdot (\sum_{q=1}^{\infty} |d_q|)^{k/n},$$

and it follows, from (94), that

$$(95) \qquad |A^{(n)}(x)| \leqq (2\pi)^n \Big\{ 2\,(\frac{|d_0|}{2} + \sum_{q=1}^{\infty} |d_q|) + \sum_{k=1}^{n-1} C_n^k \,(\sum_{k=1}^{\infty} |d_q|) \Big\} (\sum_{q=1}^{\infty} |d_q| q^n) \leqq C(4\pi)^n \sum_{q=1}^{\infty} |d_q| q^n.$$

From integration by parts of (93) repeated $l(l \geqq 3)$ times and from (91) it follows that

$$(96) \qquad (2\pi q)^l |d_q| \leqq 2 \cdot e^{2l} M'_{l-2}.$$

And finally (95) and (96) give (in putting in (96), $l = n+2$),

$$|A^{(n)}(x)| \leqq 2C(4\pi)^n (2\pi)^{-n-2} e^{2(n+2)} \left(\sum_{q=1}^{\infty} \frac{1}{q^2}\right) M_n' = L(2\alpha k e^2)^n M_n,$$

where L is a constant. If α is chosen such that $2\alpha k e^2 < 1$, the function

$$\varphi(x) = \frac{A(x)}{L}$$

satisfies all the desired conditions of the Lemma. Thus the Lemma is proved.

It results from this Lemma that, if we suppose that the class $C_{\{M_n\}}$ is not quasi-analytic, there exists, in $[0, 1]$, a function $f(x)$, belonging to $C_{\{M_n\}}$, not identically zero, not negative and such that $f^{(n)}(0)=f^{(n)}(1)=0$. We can even state that such a function exists with $|f^{(n)}(x)|<M_n (0 \leqq x \leqq 1)$ $(n \geqq 1)$. Consider now the function

$$(97) \qquad F(z)=\int_0^1 f(x)e^{-xz}\,dx.$$

$F(z)$ is an entire function of the complex variable z, with

$$(98) \qquad F(1)=\int_0^1 f(x)e^{-x}\,dx>0.$$

If $0<p<1$, let us denote by C_p the circle

$$(99) \qquad \left|\frac{1-z}{z}\right|=p.$$

The point $z=1$ is inside of every C_p. The center of C_p is the point $z_p=\frac{1}{1-p^2}$, its radius is $R_p=\frac{p}{1-p^2}$. We shall choose only such values of p that $F(z)$ has no zeros on the circumference C_p. There exists obviously an infinity of such p, with $p \nrightarrow 1$, since if this were not true, the zeros of $F(z)$ would have a limit point on a finite distance, and $F(z)$ would be identically zero, contrary to (98). Let $\alpha_1, \alpha_2 \cdots \alpha_{n(p)}$ be zeros of $F(z)$ in C_p. Then we can write

$$(100) \qquad F(z)=\Phi_p(z)(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_{n(p)})^{m_{n(p)}},$$

where $\Phi_p(z)$ is a function holomorphic in C_p (closed), without zeros in this closed circle, and where the m_j are positive integers.

Denoting by ρ_p the quantity $|z_p-1|$, since the argument of $1-z_p$ is π, we have, by Poisson integral formula,

$$\log|\Phi(1)|=\frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(z_p+R_pe^{i\varphi})|\frac{R_p^2-\rho_p^2}{R_p^2+2R_p\rho_p\cos\varphi+\rho_p^2}\,d\varphi$$

$$=\frac{p}{2\pi}\oint_{C_p}\log|\Phi(z)|\frac{|dz|}{|1-z|^2},$$

and by (99),

$$(101)\qquad \log\Phi(1)=\frac{1}{2p\pi}\oint_{C_p}\log|\Phi(z)|\frac{|dz|}{|z|^2}.$$

We have, on the other hand, in putting $z=\frac{1}{1+\zeta}$, α being in C_p,

$$\oint_{C_p}\frac{\log|z-\alpha|}{|z|^2}|dz|=\oint_{|\zeta|=p}\log\left|\frac{1}{\zeta+1}-\alpha\right||d\zeta|$$
$$=\oint_{|\zeta|=p}\log\left|\frac{1-\alpha}{\alpha}-\zeta\right||d\zeta|+\oint_{|\zeta|=p}\log|\alpha||d\zeta|-$$
$$\oint_{|\zeta|=p}\log|\zeta+1||d\zeta|.$$

But, since α is in C_p, $\left|\frac{1-\alpha}{\alpha}\right|<p$ and it is easy to calculate the first integral,[1] which is equal to $2\pi p\log p$. The last integral is zero (by Poisson integral the value of this integral is $\log|\zeta+1|$, with $\zeta=0$). Thus

$$(102)\qquad \oint_{C_p}\frac{\log|z-\alpha|}{|z|^2}|dz|=2\pi(p\log p+p\log|\alpha|)>2\pi p$$
$$\log|1-\alpha|.$$

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[1]We have $\oint_{|\zeta|=p}\log\left|\frac{1-\alpha}{\alpha}-\zeta\right||d\zeta|=p\int_0^{2\pi}\log\left|\frac{1-\alpha}{\alpha}-\zeta\right|d\varphi=2\pi p\ \log\ p+p\int_0^{2\pi}\log$ $\left|\frac{1-\alpha}{p\alpha}-e^{i\varphi}\right|d\varphi$, $(\zeta=pe^{i\varphi})$; put $\gamma=\frac{1-\alpha}{p\alpha}=be^{i\beta}$. The value of the last integral does not change if we replace γ by its absolute value b (we need only replace $\varphi-\beta$ by a new variable). But $\frac{d}{db}\int_0^{2\pi}\log|b-e^{i\varphi}|\,d\varphi=\int_0^{2\pi}\frac{b-\cos\varphi}{1-2b\cos\varphi+b^2}\,d\varphi=0$, and $\int_0^{2\pi}\log|$ $e^{i\varphi}|\,d\varphi=0$; thus $\int_0^{2\pi}\log|b-e^{i\varphi}|\,d\varphi=0$.

The formulas (100), (101) and (102) give thus immediately

(103) $$\log|F(1)| = \log|\Phi_p(1)| + m_1 \log|1-\alpha_1| + \cdots + m_n \log|1-\alpha_n| \leqq \frac{1}{2p\pi}\oint_{C_p} \log|\Phi_p(z)|\frac{|dz|}{|z|^2} + \frac{m_1}{2p\pi}\oint_{C_p}\frac{\log|z-\alpha_1|}{|z|^2}|dz| + \cdots + \frac{m_n}{2p\pi}\oint_{C_p}\frac{\log|z-\alpha_n|}{|z|^2}|dz| = \frac{1}{2p\pi}\oint_{C_p}\log|F(z)|\frac{|dz|}{|z|^2}.$$

Integrating by parts (97), k times, we get, since

$$f^{(n)}(0) = f^{(n)}(1) = 0 (n \geqq 0),$$

$$F(z) = \frac{1}{z^k}\int_0^1 f^{(k)}(z)e^{-xz}\,dx.$$

Therefore for $Rz \geqq 0$, we have the inequality

(104) $$|F(z)| \leqq \frac{1}{|z|^k}\int_0^1 |f^{(k)}(x)|\,dx \leqq \frac{M_k}{|z|^k}.$$

Since this inequality is satisfied for every k, we have also if $\lim M_n^{1/n} = \infty$,

$$|F(z)| \leqq \operatorname*{Min}_{k\geqq 0}\frac{M_k}{|z|^k} = \frac{1}{\operatorname{Max}\frac{|z|^k}{M_k}} = \frac{1}{T(|z|)}.$$

We have therefore, by (103),

$$\frac{1}{2p\pi}\oint_{C_p}\frac{\log T(|z|)}{|z|^2}|dz| \leqq -\frac{1}{2p\pi}\oint_{C_p}\frac{\log|F(z)|}{|z|^2}|dz| \leqq -\log|F(1)|.$$

If we denote by C_p^t the part of the circle C_p, passing through the point $Z = \frac{1}{p+1}$, and contained between the two lines $Iz = -t$, $Iz = t$,[1] we have, for large values of t (since $\log T(|z|)$ tends to infinity with $|z|$),

$$\frac{1}{2\pi p}\int_{C_p^t}\frac{\log T(|z|)}{|z|^2}|dz| \leqq -\log|F(1)|.$$

[1] Iz denotes the coefficient of i in z.

But when p tends to unity, C_p^t tends to the segment of the straight line $Rz=\frac{1}{2}$, situated between the two lines $Iz=-t$, and $Iz=t$. Therefore, if t is positive large,

$$(105)\qquad \frac{1}{\pi}\int_0^t \frac{\log T(r)}{\frac{1}{4}+r^2}\,dr \leqq \frac{1}{2\pi}\int_{-t}^t \frac{\log T(|\frac{1}{2}+ir|)}{|\frac{1}{2}+ir|^2}\,dr$$
$$\leqq -\log|F(1)|,$$

which proves that the integral

$$\int_1^\infty \frac{\log T(r)}{r^2}\,dr$$

converges. Therefore, if this integral diverges, $C_{\{M_n\}}$ is a quasi-analytic class.

If we had $\underline{\lim}\, M_n^{1/n}<\infty$, $\underset{k\geqq 1}{\mathrm{Bd}}\, \frac{M_k}{|z|^k}$ would be, for large values of $|z|$ zero, and, by (104), $F(z)$ would be zero for large values of $|z|$, i.e., identically zero, contrary to (98). Thus, in the case where $\underline{\lim}\, M_n^{1/n}<\infty$, the class is also quasi-analytic. Our theorem is thus completely proved.

Remark. We shall notice that from (105) it follows that, if in $[0,\ 1]$, $f(x)\geqq 0$; $f^{(n)}(0)=f^{(n)}(1)=0$, $(n\geqq 0)$, $f(x)\not\equiv 0$ $|f^{(n)}(x)|<M_n$ $(n\geqq 1)$, then

$$\int_0^\infty \frac{\log T(r)}{\frac{1}{4}+r^2}\,dr \leqq -\pi\log\int_0^1 f(x)e^{-x}dx,$$

and, since $T(r)\geqq \frac{r}{M_1}$, we have

$$\int_0^1 \frac{\log r\,dr}{\frac{1}{4}+r^2}-\int_0^1 \frac{\log M_1 dr}{\frac{1}{4}+r^2}+\int_1^\infty \frac{\log T(r)}{\frac{1}{4}+r^2}\,dr \leqq -\pi\log\int_0^1 f(x)e^{-x}dx,$$

which gives, by Lemma IV[1], since for $r\geqq 1: \frac{1}{4}+r^2\leqq \frac{5}{4}r^2$,

$$\sum_1^\infty \frac{M_n^c}{M_{n+1}^c}=\int_1^\infty \frac{\log T(r)}{r^2}\,dr-\log T(1)-1\leqq \frac{5}{4}\int_1^\infty \frac{\log T(r)}{\frac{1}{4}+r^2}\,dr+$$

$$\log M_1 - 1 \leqq \log M_1(1+\tfrac{5}{2} \text{ arc tang } 2) - \frac{5\pi}{4} \log \int_0^1 f(x)e^{-x}dx + 4.$$

For instance, if $M_n = \text{Max } |f^{(n)}(x)| (n \geqq 1)$, $M_1 = 1$, $f^{(n)}(0) = f^{(n)}(1) = 0 (n \geqq 0)$, $f(x) \not\equiv 0$,

$$\sum_1^\infty \frac{M_n^c}{M_{n+1}^c} \leqq 4 - \frac{5\pi}{4} \log \int_0^1 f(x)e^{-x}dx.$$

14. DECOMPOSITION OF AN INFINITELY DIFFERENTIABLE FUNCTION

We shall prove, in this paragraph, the following very general theorem proved by the author in 1939. [**12**]:

THEOREM XIII. *Every function, infinitely differentiable, in a closed interval $[a, b]$, is a sum of two infinitely differentiable functions, each belonging, in $[a, b]$, to a quasi-analytic class.*

Obviously, if an i. d. function does not belong to a quasi-analytic class, the two functions into which we decompose it by theorem XIII, do not belong to the same class, since, if they did, the given function would also belong to the same class, contrary to the hypothesis.

We may suppose that $[a, b]$ is the interval $[-1, +1]$. Let then $f(x)$ be an i. d. function in $[-1, +1]$. We shall suppose that $f(x)$ is not a polynomial, since for a polynomial the theorem is obvious. Then, by Lemma VI, we have, in $[-1, +1]$,

$$f(x) = \sum a_n T_n(x),$$

with

$$|a_n| < \frac{1}{S\left(\frac{n}{A}\right)} \quad (n \geqq 1), \tag{106}$$

where

$$S(r) = \operatorname*{Max}_{r \geqq p} \frac{r^p}{m_p},$$

$$0 < m_p = \max_{-1 \leqq x \leqq 1} |f^{(p)}(x)| \, (p \geqq 1),$$

and where A is a positive constant.

By (35) we have, if $r > r_0 > 0$,

$$\log S(r) = \log S(r_0) + \int_{r_0}^{r} \frac{M(t)}{t}\, dt + \int_{r_0}^{r} dW(t),$$

where $M(t)$ is a positive function increasing to infinity with t, and taking integral values; the second integral is positive.

Let us choose $r_0 > 0$ such that $M(r_0) > 2$, $\log S(r_0) > 0$, and let us put $m(r) = M(r) - 2$, if $r \geqq r_0$, and $m(r) = 0$, if $r < r_0$.

We have the following inequality for $r \geqq r_0$:

$$\log S(r) \geqq 2 \log\left(\frac{r}{r_0}\right) + \int_0^r \frac{m(t)}{t}\, dt = 2 \log\left(\frac{r}{r_0}\right) + \int_0^{r^2} \frac{m(t^{1/2})}{2t}\, dt,$$

and putting

$$N(t) = \left[\frac{m(t^{1/2})}{2}\right],$$

we have, for $r \geqq r_0$,

$$\log S(r) \geqq 2 \log\left(\frac{r}{r_0}\right) + \int_0^{r^2} \frac{N(t)}{t}\, dt, \tag{107}$$

where $N(t)$ is a non-negative function, taking integral values, and tending increasingly to infinity with t. $N(t)$ is obviously zero for small positive values of t.

The quantity a being greater than unity, denote by $\{t_q\}\,(q \geqq 0)$ a sequence satisfying the following inequalities:

$$t_1 > t_0 = 0,\ t_{q+1} > a t_q (q \geqq 1),$$

$$\left[\frac{1}{\log\left(\frac{t_q}{t_{q-1}}\right)} \int_{t_{q-1}}^{t_q} \frac{N(t)}{t}\, dt\right] > t_{q-1},\ (q \geqq 2). \tag{108}$$

Such a sequence $\{t_q\}$ exists. Indeed, if α, β and λ are positive quantities, with $\lambda > e$, $\beta > \lambda\alpha > 0$, the following inequalities are satisfied

$$\int_\alpha^\beta \frac{N(x)}{x}\,dx > \int_{\lambda\alpha}^\beta \frac{N(x)}{x}\,dx \geqq N(\lambda\alpha)\log\left(\frac{\beta}{\lambda\alpha}\right) =$$
$$N(\lambda\alpha)\log\left(\frac{\beta}{\alpha}\right)\left(1-\frac{\log\lambda}{\log(\beta/\alpha)}\right),$$

and if, α being given, we choose β and λ such that

$$\frac{\log\lambda}{\log(\beta/\alpha)} < \tfrac{1}{2},$$

then

$$\frac{1}{\log(\beta/\alpha)}\int_\alpha^\beta \frac{N(x)}{x}\,dx \geqq \tfrac{1}{2}N(\lambda\alpha).$$

Thus, when α is fixed, the left-hand expression tends to infinity, with β.

Let us now put

$$N_q = \left[\frac{1}{\log\left(\dfrac{t_q}{t_{q-1}}\right)}\int_{t_{q-1}}^{t_q}\frac{N(t)}{t}\,dt\right] \quad (q \geqq 2),$$

and

$$N_1(r) = \left\{\begin{array}{l} N(r), \text{ if } t_{2q} \leqq r < t_{2q+1} \\ N_{2(q+1)}, \text{ if } t_{2q+1} \leqq r < t_{2(q+1)} \end{array}\right\} (q \geqq 0),$$

$$N_2(r) = \begin{cases} N(r), \text{ if } t_{2q+1} \leqq r < t_{2(q+1)},\ (q \geqq 0), \text{ and if } 0 \leqq r < t_1 \\ N_{2q+1}, \text{ if } t_{2q} \leqq r < t_{2q+1} (q \geqq 1). \end{cases}$$

Let us also introduce the functions $T(r)$, $T_1{}^*(r)$, $T_2{}^*(r)$, defined in the following manner:

$$\log T(r) = \int_0^r \frac{N(t)}{t}\,dt,$$

(109)
$$\log T_1^*(r) = \int_0^r \frac{N_1(t)}{t} dt,$$
$$\log T_2^*(r) = \int_0^r \frac{N_2(t)}{t}\, dt.$$

The functions $N_1(r)$, $N_2(t)$ have, both, the value zero for small values of r, they are integral valued, and they increase to infinity with r. Let us prove these properties for $N_1(r)$. The proof is the same for $N_2(r)$. The first two properties are evident by definition ($N(r)=0$ for small values of r). In $[t_{2q}, t_{2q+1})$, $N_1(r)$ increases, by definition. In $[t_{2q+1}, t_{2(q+1)})$, we have, with $t_{2q} \leqq \rho < t_{2q+1}$,

$$N_1(r) = N_1(t_{2q+1}) = N_{2(q+1)} = \left[\frac{1}{\log\left(\dfrac{t_{2(q+1)}}{t_{2q+1}}\right)} \int_{t_{2q+1}}^{t_{2(q+1)}} \frac{N(t)}{t}\, dt\right]$$
$$\geqq \left[N(t_{2q+1}) \frac{1}{\log\left(\dfrac{t_{2(q+1)}}{t_{2q+1}}\right)} \int_{t_{2q+1}}^{t_{2(q+1)}} \frac{dt}{t}\right] = [N(t_{2q+1})] = N(t_{2q+1})$$
$$\geqq N(\rho) = N_1(\rho).$$

Thus, $N_1(r)$ increases. But, since $N_1(t_{2q}) = N(t_{2q})$ tends to infinity with q, $N_1(r)$ tends increasingly to infinity.

We have, in each interval $[t_{2q}, t_{2q+1})$, the inequality
$$T_1^*(r) \leqq T(r),$$
and, in each interval $[t_{2q+1}, t_{2(q+1)})$ and in $[0, t_1)$ the inequality
$$T_2^*(r) \leqq T(r).$$

Let us prove the first inequality in $[t_{2q}, t_{2q+1})$. We have, in this interval,

$$\log T_1^*(r) = \int_0^r \frac{N_1(t)}{t}\, dt = \int_0^{t_1} \frac{N(t)}{t}\, dt + N_2 \log\left(\frac{t_2}{t_1}\right) + \int_{t_2}^{t_3} \frac{N(t)}{t}\, dt$$
$$+ \cdots + N_{2q} \log\left(\frac{t_{2q}}{t_{2q-1}}\right) + \int_{t_{2q}}^{r} \frac{N(t)}{t}\, dt.$$

But, by the definition of the quantities N_q,

$$N_{2p} \log\left(\frac{t_{2p}}{t_{2p-1}}\right) \leqq \int_{t_{2p-1}}^{t_{2p}} \frac{N(t)}{t}\,dt,$$

therefore we have, in $[t_{2q}, t_{2q+1})$,

$$\log T_1^*(r) \leqq \int_0^r \frac{N(t)}{t}\,dt = \log T(r).$$

The proof of the corresponding inequality for $T_2^*(r)$ is similar.

We shall now prove that

$$(110) \qquad \int_1^\infty \frac{\log T_1^*(r)}{r^2}\,dr = \int_1^\infty \frac{\log T_2^*(r)}{r^2}\,dr = \infty.$$

Let us prove, for instance, that the first integral diverges. If $t_{2q-1} \leqq r < t_{2q}$, then

$$\log T_1^*(r) = \int_0^r \frac{N_1(t)}{t}\,dt \geqq \int_{t_{2q-1}}^r \frac{N_1(t)}{t}\,dt \geqq N_1(t_{2q-1}) \log\left(\frac{r}{t_{2q-1}}\right)$$

$$= N_{2q} \log\left(\frac{r}{t_{2q-1}}\right).$$

And, if we put now $\tau = \dfrac{r}{t_{2q-1}}$, we have by the definition of $\{t_p\}$ [(108)] for $q \geqq 1$,

$$\int_{t_{2q-1}}^{t_{2q}} \frac{\log T_1^*(r)}{r^2}\,dr \geqq N_{2q} \int_{t_{2q-1}}^{t_{2q}} \frac{\log\left(\frac{r}{t_{2q-1}}\right)}{r^2}\,dr = \frac{N_{2q}}{t_{2q-1}} \int_1^{\frac{t_{2q}}{t_{2q-1}}} \frac{\log \tau}{\tau^2}\,d\tau$$

$$\geqq \frac{N_{2q}}{t_{2q-1}} \int_1^a \frac{\log \tau}{\tau^2}\,d\tau \geqq \int_1^a \frac{\log \tau}{\tau^2}\,d\tau = b > 0,$$

where b is a positive constant independent of q. Thus

$$\sum_{q=1}^\infty \int_{t_{2q-1}}^{t_{2q}} \frac{\log T_1^*(r)\,dr}{r^2} \leqq \int_{t_1}^\infty \frac{\log T_1^*(r)}{r^2}\,dr = \infty.$$

Let us now define quantities b_n and c_n in the following manner (the quantities a_n, and A were defined previously):

$$b_n = \begin{cases} a_n, \text{ if } A\sqrt{t_{2q}} \leqq n < A\sqrt{t_{2q+1}} \\ 0\ , \text{ if } A\sqrt{t_{2q+1}} \leqq n < A\sqrt{t_{2(q+1)}} \end{cases} (q \geqq 0),$$

(111)

$$c_n = \begin{cases} a_n, \text{ if } A\sqrt{t_{2q+1}} \leqq n < A\sqrt{t_{2(q+1)}} \\ 0\ , \text{ if } A\sqrt{t_{2q}} \leqq n < A\sqrt{t_{2q+1}} \end{cases} (q \geqq 0).$$

We have obviously for $n \geqq 0$,

$$b_n + c_n = a_n.$$

Let us now put

$$f_1(x) = \sum_{n=0}^{\infty} b_n T_n(x),$$

$$f_2(x) = \sum_{n=0}^{\infty} c_n T_n(x).$$

Thus we have in $[-1, +1]$,

$$f(x) = f_1(x) + f_2(x).$$

We shall now show that each function $f_1(x)$, $f_2(x)$ belongs to a corresponding quasi-analytic class.

From (106), (107), (109), (111), and from the inequality $T_1^*(r) \leqq T(r)$, satisfied in $[t_{2q}, t_{2q+1})$, it follows that for $n \geqq 1$,

$$|b_n| \leqq \frac{1}{S\left(\dfrac{n}{A}\right)} \leqq \left(\frac{Ar_0}{n}\right)^2 \frac{1}{T\left(\dfrac{n^2}{A^2}\right)} \leqq \left(\frac{Ar_0}{n}\right)^2 \frac{1}{T_1^*\left(\dfrac{n^2}{A^2}\right)}.$$

It follows from this inequality and from the inequality

$$p^p |T_n^{(p)}(x)| \leqq \left(\frac{e}{2}\right)^p n^{2p}, \quad [-1 \leqq x \leqq 1], \quad (1 \leqq p \leqq n)$$

that in $[-1, +1]$; for $p \geqq 1$,

$$|f_1^{(p)}(x)| \leqq \sum_{n \geqq p}^{\infty} |b_n|\,|T_n^{(p)}(x)| \leqq (Ar_0)^2 \left(\frac{e}{2p}\right)^p \sum_{n \geqq p}^{\infty} \frac{n^{2p}}{n^2 T_1^*\left(\dfrac{n^2}{A^2}\right)}$$

$$\leqq (Ar_0)^2 \left(\frac{e}{2p}\right)^p \left(\sum_1^{\infty} \frac{1}{n^2}\right) \operatorname*{Max}_{n \geqq p} \frac{n^{2p}}{T_1^*\left(\dfrac{n^2}{A^2}\right)} \leqq C^p \operatorname*{Max}_{n \geqq p} \frac{\left(\dfrac{n}{A}\right)^{2p}}{T_1^*\left\{\left(\dfrac{n}{A}\right)^2\right\}}$$

$$\leqq C^p \operatorname*{Max}_{r \geqq 0} \frac{r^p}{T_1^*(r)}.$$

If we put

$$M'_p = \operatorname*{Max}_{r \geqq 0} \frac{r^p}{T_1^*(r)} \quad (p \geqq 1), \tag{112}$$

the function $f_1(x)$ belongs to the class $C_{\{M_p'\}}$.

But from (112) it follows that for sufficiently large r

$$T_1^*(r) = \operatorname*{Max}_{p \geqq 1} \frac{r^p}{M_p'}.$$

Indeed the inequality

$$T_1^*(r) \geqq \operatorname*{Max}_{p \geqq 1} \frac{r^p}{M_p'} \tag{113}$$

follows from (112). But one can also write, if $N_1(r) > 1$,

$$\operatorname*{Max}_{p \geqq 1} \frac{r^p}{M_p'} = \operatorname*{Max}_{p \geqq 1} \frac{r^p}{\operatorname*{Max}\limits_{\rho \geqq 0} \dfrac{\rho^p}{T_1^*(\rho)}} \geqq \frac{r^{N_1(r)}}{\operatorname*{Max}\limits_{\rho \geqq 0} \dfrac{\rho^{N_1(r)}}{T_1^*(\rho)}}. \tag{114}$$

Since, for $\rho_2 > \rho_1$, we have

$$\log\left(\frac{\rho_2^{N_1(r)}}{T_1^*(\rho_2)}\right) - \log\left(\frac{\rho_1^{N_1(r)}}{T_1^*(\rho_1)}\right) = N_1(r)\log\rho_2 - \int_0^{\rho_2} \frac{N_1(t)}{t}\,dt$$
$$-\left(N_1(r)\log\rho_1 - \int_0^{\rho_1} \frac{N_1(t)}{t}\,dt\right) = N_1(r)\log\left(\frac{\rho_2}{\rho_1}\right) - \int_{\rho_1}^{\rho_2} \frac{N_1(t)}{t}\,dt,$$

that is to say,

$$\Big(N_1(r) - N_1(\rho_2)\Big)\log\left(\frac{\rho_2}{\rho_1}\right) \leqq \log\left(\frac{\rho_2^{N_1(r)}}{T_1^*(\rho_2)}\right) - \log\left(\frac{\rho_1^{N_1(r)}}{T_1^*(\rho_1)}\right)$$
$$\leqq \Big(N_1(r) - N_1(\rho_1)\Big)\log\left(\frac{\rho_2}{\rho_1}\right),$$

we see that the expression

$$\frac{\rho^{N_1(r)}}{T_1^*(\rho)}$$

has the same value (when r is fixed) for all values ρ such that

$$N_1(\rho) = N_1(r),$$

that it increases when $N_1(\rho) < N_1(r)$, and decreases when $N_1(\rho) > N_1(r)$.

Therefore we have

$$\operatorname*{Max}_{\rho \geqq 0} \frac{\rho^{N_1(r)}}{T_1^*(\rho)} = \frac{r^{N_1(r)}}{T_1^*(r)},$$

and it follows from (114) that, for r such that $N_1(r) > 1$,

$$\operatorname*{Max}_{p \geqq 1} \frac{r^p}{M_p'} \geqq T_1^*(r),$$

which, together with (113), gives, for r such that $N_1(r) > 1$,

$$T_1^*(r) = \operatorname*{Max}_{p \geqq 1} \frac{r^p}{M_p'}. \tag{115}$$

From (110) and from the fundamental Theorem XII, on quasi-analytic classes, it follows that $C_{\{M'_n\}}$ is quasi-analytic.

One can prove, in the same manner, that $f_2(x)$ belongs to the class $C_{\{M_n^{(2)}\}}$,where

$$M_n^{(2)} = \operatorname*{Max}_{r \geqq 0} \frac{r^n}{T_2^*(r)},$$

and that this class is quasi-analytic. Our theorem is thus proved.

Remark. From (112) and (115) it follows that for large values of n: $M_n'^c = M_n$, and from the quasi-analyticity of $C_{\{M_n'\}}$ it follows that

$$\sum \frac{M_n'^c}{M_{n+1}'^c} = \sum \frac{M_n'}{M_{n+1}'} = \infty. \tag{116}$$

Let Π be the ∞-base of $\{\log M_n'\}$. If α is the quantity such that there exists a point $(n, \log M_n')$ on the straight line $y = \alpha x$, and there is no such point below this line, then, for n such that $\frac{\log M_n'}{n} > \alpha$, the following inequality is satisfied:

$$\log M_{n+1}' - \log M_n' = \log M_{n+1}'^c - \log M_n^c \geqq \frac{\log M_n'^c}{n} = \frac{\log M_n'}{n}.$$

In other words, we have

$$(M_n')^{1/n} \leqq \frac{M_{n+1}'}{M_n'}.$$

And it follows from (116) that

$$\sum \frac{1}{(M_n')^{1/n}} = \infty.$$

We have obviously also

$$\sum \frac{1}{(M_n^{(2)})^{1/n}} = \infty.$$

Carleman (**2**) proposed, in 1926, the following question: Let $f_1(x)$ and $f_2(x)$ be two i. d. functions in an interval $[a, b]$, such that, if we put

$$m_n' = \text{Max } |f_1^{(n)}(x)| \quad (n \geqq 0)$$
$$m_n^{(2)} = \text{Max } |f_2^{(n)}(x)| \quad (n \geqq 0),$$

we have

(117) $$\sum \frac{1}{(m_n')^{1/n}} = \sum \frac{1}{(m_n^{(2)})^{1/n}} = \infty,$$

and such that

(117^1) $$f_1^{(n)}(a) = f_2^{(n)}(a) \quad (n \geqq 0).$$

Can one say that, in $[a, b]$, the two functions are identical?

San Juan (**18**) gave to this question a negative answer. He proved that the function

$$\varphi(x) = \int_0^\infty e^{-\sqrt[4]{t}} (\sin \sqrt[4]{t})\, e^{-xt} dt \quad (x \geqq 0),$$

which is such that $\varphi^{(n)}(0) = 0$ $(n \geqq 0)$, is a sum of two functions $f_1(x)$ and $f_2(x)$ for which (117) is satisfied. Therefore the two functions $f_1(x)$ and $-f_2(x)$ obtained in this fashion give a negative answer to Carleman's question.

But the general Theorem XIII which we proved in this paragraph gives, in particular, a much larger answer to Carleman's question. Indeed, it follows from Theorem XIII and from the remark which follows its proof, applied to functions which are zero at a, and of which the derivatives are zero at this point, that

Every i. d. function in an interval $[a, b]$, which is equal to zero at a and of which the derivatives are zero at this point, is the difference of two i. d. functions $f_1(x)$, $f_2(x)$ satisfying (117) *and* (117^1).

15. APPLICATIONS TO INTEGRABLE FUNCTIONS NEW QUASI-ANALYTIC CLASSES

We shall see that some of the statements on quasi-analytic classes might be applied to the study of functions integrable L. The magnitude of an integrable function $f(x)$ (by integrable we shall always understand integrable L) in the neighborhood of a point x_0 can be only characterized by the value of the integral

$$\int_{x_0}^{x_0+\alpha} |f(x)|\, dx.$$

Thus, we shall introduce the following definition:

If $f(x)$ is integrable in $[a, b]$, and if for a point $x_0 (a \leqq x_0 < b)$ we have

$$\varlimsup_{\alpha=+0} \frac{\log\left(-\log \int_{x_0}^{x_0+\alpha} |f(t)|\, dt\right)}{-\log \alpha} = \rho,$$

then we shall say that x_0 is, for $f(x)$, *a right-hand mean-value zero of exponential order* ρ.

If for $a < x_0 \leqq b$ we have

$$\varlimsup_{\alpha=-0} \frac{\log\left(-\log \int_{x_0+\alpha}^{x_0} |f(t)|\, dt\right)}{-\log(-\alpha)} = \rho,$$

then x_0 is, for $f(x)$, a *left-hand mean-value zero of exponential order* ρ.

It is easy to verify that, if $f(x)$ is continuous at x_0, and is, in a neighborhood of this point, of the form

$$f(x) = \varphi(x)(x - x_0)^k,$$

where $\varphi(x)$ is continuous at x_0, $\varphi(x_0) \neq 0$, then $\rho = 0$. But if, in such a neighborhood, we have

$$f(x) = \varphi(x) e^{-\frac{1}{(x-x_0)^k}},$$

then $\rho = k$.

Now we shall recall the definition of the index of convergence of a sequence of positive numbers.

The index of convergence γ, of the sequence $\{n_i\}$ $(n_i \uparrow \infty)$ is the greatest lower bound of the quantities k such that

$$\sum_{i=\infty} \frac{1}{n_i^k} < \infty.$$

Thus for any $\epsilon > 0$, we have

$$\sum \frac{1}{n_i^{\gamma+\epsilon}} < \infty,$$

$$\sum \frac{1}{n_i^{\gamma-\epsilon}} = \infty.$$

The problem we are dealing with in this paragraph is the following one:

To give conditions such that, if the Fourier coefficients of an integrable function in $[0, 2\pi]$, *which has a mean-value zero of exponential order in* $[a, b]$ *satisfies them, then this function is zero almost everywhere.*

One answer to this question is given by the following theorem:

THEOREM XIV. *Let* $f(x)$ *be integrable in* $[0, 2\pi]$, *and suppose that* $x = x_0$ $(x_0 \epsilon [0, 2\pi])$ *is for* $f(x)$ *a right-hand (or a left-hand) mean-value zero of exponential order* $\delta > 0$.

Suppose that the Fourier series of $f(x)$ *is of the form*

$$f(x) \backsim \sum_{i=1} (a_{m_i} \cos m_i x + b_{m_i} \sin m_i x),$$

the index of convergence of $\{m_i\}$ *being* σ *with* $\sigma < 1$. *If*

$$\delta > \frac{\sigma}{1-\sigma},$$

then $f(x)$ *is zero almost everywhere.*

This theorem will allow us later on to introduce new and important quasi-analytic classes.

The proof of this theorem is essentially based on the following Lemma.

LEMMA IX. *Let* $\{n_i\}$ *be a positive increasing sequence of integers with an index of convergence* $\sigma<1$. *Let* m *be a positive integer different from all the* $n_i (i \geqq 1)$.

If a *and* b *are two real quantities such that*

$$a^2+b^2>0,$$

and if ρ *is a quantity such that*

$$\rho>\frac{\sigma}{1-\sigma},$$

then, to every positive α, *sufficiently small, there corresponds a function* $\gamma_\alpha(x)$ *having the properties:*

(1) $\gamma_\alpha(x)$ *is an i. d. function in* $[0, \alpha]$;

(2) $\gamma_\alpha^{(n)}(0)=\gamma_\alpha^{(n)}(\alpha)=0 \quad (n\geqq 0)$;

(3) *on putting*

$$a_n^{(\alpha)}=\int_0^\alpha \gamma_\alpha(x)\cos nx dx,$$

$$b_n^{(\alpha)}=\int_0^\alpha \gamma_\alpha(x)\sin nx dx,$$

we shall have

$$a_0=a_{n_i}^{(\alpha)}=b_{n_i}^{(\alpha)}=0;$$

(4) *there exists a positive constant* ω, *independent of* α *such that*

$$|aa_m^{(\alpha)}+bb_m^{(\alpha)}|>\omega\alpha^2;$$

(5) $$|\gamma_\alpha(x)|<e^{\alpha^{-\rho}}.$$

Let us introduce three positive constants η_1, η_2, η_3 satisfying the inequality

(118) $$\gamma=\eta_1+\eta_2+\eta_3<\frac{1-\sigma}{\sigma}-\frac{1}{\rho}.$$

Let us now write

$$M_n=\left(\frac{n}{2}\right)^{n(1+\eta_1)}.$$

We have

$$T(r)=\operatorname*{Max}_{n\geqq 1}\frac{r^n}{M_n}\leqq \operatorname*{Max}_{x\geqq 0}\frac{r^x}{\left(\dfrac{x}{2}\right)^{x(1+\eta_1)}}=e^{\frac{2(1+\eta_1)}{e}\cdot r^{\frac{1}{1+\eta_1}}}.$$

Thus

$$\int_1^\infty \frac{\log T(r)}{r^2}\, dr \leqq \frac{2(1+\eta_1)}{e} \int_1^\infty \frac{dr}{r^{2-\frac{1}{1+\eta_1}}} < \infty,$$

and the class $C_{\{M_n\}}$ is not quasi-analytic. It is then possible, by Lemma VIII, to construct a function $\varphi(x)$, i. d. in $[0, 1]$, and having the following properties:

(a) $\varphi(x) \not\equiv 0$

(b) $\varphi(x) \geqq 0$

(c) $\varphi(x) = \varphi(1-x)$

(d) $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0, \quad (n \geqq 0),$

(e) $|\varphi^{(n)}(x)| < M_n.$

If we write

$$\varphi_\alpha(x) = \varphi\left(\frac{x}{\alpha}\right),$$

we have obviously in $[0, \alpha]$ the inequality

(119) $$|\varphi_\alpha^{(n)}(x)| \leqq \frac{M_n}{\alpha^n} = \left(\frac{n^{1+\eta_1}}{\alpha 2^{1+\eta_1}}\right)^n.$$

We shall consider now the entire function

$$F(z) = z^2 \prod_1^\infty \left(1 - \frac{z^2}{n_i^2}\right) = \sum_{n=1}^\infty C_{2n} z^{2n}.$$

It is well known, by the classical theory of entire functions (**20**) that, if the index of convergence of $\{n_i\}$ is σ, then the order of $F(z)$ is σ; in other words, if we put

$$M(r) = \text{Max}\, |F(re^{i\theta})| \quad (0 \leqq \theta \leqq 2\pi),$$

we have

$$\overline{\lim_{r=\infty}} \frac{\log(\log M(r))}{\log r} = \sigma.$$

And, since $F(z)$ is of order σ, it is also well known that

$$\underline{\lim} \frac{-\log|C_{2n}|}{2n \log n} = \frac{1}{\sigma}.$$

There exists therefore a constant A_1 such that for $n \geqq 1$,

(120) $$|C_{2n}| < \frac{A_1}{n^{2n(1/\sigma - \eta_2)}}.$$

We shall study the function

$$\psi_\alpha(x)=\sum_{n=1}^{\infty}(-1)^n C_{2n}\varphi_\alpha^{(2n)}(x).$$

From (119) and (120) it follows that in $[0, \alpha]$,

$$|\psi_\alpha(x)|\leqq A_1\sum_{n=1}^{\infty}\left(\frac{(2n)^{1+\eta_1}}{\alpha\cdot 2^{1+\eta_1}}\right)^{2n} n^{-2n(1/\sigma-\eta_2)},$$

and by (118) we get

(121) $$|\psi_\alpha(x)|\leqq A_1\sum_{n=1}^{\infty}\left(\frac{n^{1-1/\sigma+\gamma}}{\alpha}\right)^{2n} n^{-2n\eta_3}.$$

By (118) we have $\dfrac{\sigma-1}{\sigma}+\gamma=-\dfrac{1}{\rho_1}<-\dfrac{1}{\rho}$ and

$$\left(\frac{n^{1-1/\sigma+\gamma}}{\alpha}\right)^{2n}=\left(\frac{1}{\alpha n^{1/\rho_1}}\right)^{2n}\leqq\frac{1}{\min\limits_{n\geqq 1}(\alpha n^{1/\rho_1})^{2n}}\leqq\frac{1}{\min\limits_{x\geqq 0}(\alpha x^{1/\rho_1})^{2x}}$$
$$=e^{(2/e\rho_1)\,\alpha^{-\rho_1}}=e^{A_3\alpha^{-\rho_1}}.$$

Thus, by (119) we shall have

$$|\psi_\alpha(x)|<A_1\left(\sum_{n=1}^{\infty}n^{-2n\eta_3}\right)e^{A_3\alpha^{-\rho_1}}=A_2 e^{A_3\alpha^{-\rho_1}}.$$

For small values of α $(\alpha>0)$ we have

$$A_3\alpha^{-\rho_1}<\alpha^{-\rho}.$$

Therefore, for small α, we have

(122) $$|\psi_\alpha(x)|<A_2 e^{\alpha^{-\rho}}.$$

We shall now prove that the desired functions $\gamma_\alpha(x)$ are given by the equality

$$\gamma_\alpha(x)=\frac{\psi_\alpha(x)}{A_2}.$$

The inequality (122) proves that the condition (5) of the Lemma is satisfied.

In order to prove that the other required properties are satisfied, let us consider the Fourier series of the function $\bar{\varphi}_\alpha(x)$ defined in $[0, 2\pi]$ by the equalities

$$\bar{\varphi}_\alpha(x)=\begin{cases}\varphi_\alpha(x) & \text{if } 0\leqq x\leqq\alpha\\ 0 & \text{if } \alpha\leqq x\leqq 2\pi\end{cases}.$$

Since $\varphi_\alpha^{(n)}(\alpha)=0(n\geqq 0)$, $\bar{\varphi}_\alpha(x)$ is an i. d. function. Therefore

$$\bar{\varphi}_\alpha(x)=\frac{l_0^{(\alpha)}}{2}+\sum_{n=1}^{\infty}(l_n^{(\alpha)}\cos nx+p_n^{(\alpha)}\sin nx),$$

the coefficients $l_n^{(\alpha)}$, $p_n^{(\alpha)}$ being given by the equalities (obtained by integrating by parts $2p$ times, the Fourier formula for the coefficients)

$$l_n^{(\alpha)}=\pm\frac{1}{\pi n^{2p}}\int_0^\alpha \varphi_\alpha^{(2p)}(x)\cos nxdx\ (p\geqq 0).$$

It follows from (119) that for $\alpha<\pi$,

$$|l_n^{(\alpha)}|\leqq\left(\frac{p^{1+\eta_1}}{n\alpha}\right)^{2p}(p\geqq 0),$$

and for $n>n_\alpha$

$$(123)\qquad |l_n^{(\alpha)}|\leqq\operatorname*{Min}_{p\geqq 0}\left(\frac{p^{1+\eta_1}}{n\alpha}\right)^{2p}\leqq e^{-A_4(\alpha n)^{1/(1+\eta_1)}},\ (A_4>0).$$

We have a similar inequality for $p_n^{(\alpha)}$.

On the other hand, we have seen that $F(z)$ is of order σ, that is to say, to every $\epsilon>0$, corresponds a constant, say A_5, such that

$$M(r)<A_5e^{r^{\sigma+\epsilon}}$$

and, since

$$|F(in)|\leqq M(n),$$

we have

$$|F(in)|=\sum_{k=1}^{\infty}|C_{2n}|n^{2k}<A_5e^{n^{\sigma+\epsilon}}$$

and by (123)

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|C_{2n}|n^{2k}(|l_n^{(\alpha)}|+|p_n^\alpha|)<A_6^{(\alpha)}\sum_{n=1}^{\infty}e^{n^{\sigma+\epsilon}-A_4(\alpha n)^{1/(1+\eta_1)}}.$$

If we choose $0<\epsilon<\frac{1}{1+\eta_1}-\sigma$(this is possible since $\eta_1<\frac{1}{\sigma}-1$), we have, for $n>m_\alpha$,

$$n^{\sigma+\epsilon}-A_4(\alpha n)^{1/(1+\eta_1)}<-A^{(\alpha)}n^{1/(1+\eta_1)},$$

where $A^{(\alpha)}$ is a positive constant depending on α.

Therefore

$$(124)\quad \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|C_{2n}|n^{2k}(|l_n^{(\alpha)}|+|p_n^{(\alpha)}|)<A_7^{(\alpha)}\sum_{n=1}^{\infty}e^{-A^{(\alpha)}n^{1/(1+\eta_1)}}$$

where $A_7^{(\alpha)}$ depends also on α.

If we insert in the formula

$$(125)\qquad \overline{\psi}_\alpha(x)=\sum_{k=1}^{\infty}(-1)^k C_{2k}\overline{\varphi}_\alpha^{(2k)}(x)$$

the expansion

$$(126)\quad \overline{\varphi}_\alpha^{(2k)}(x)=(-1)^k\sum_{n=1}^{\infty}(l_n^{(\alpha)}\cos nx+p_n^{(\alpha)}\sin nx)n^{2k},$$

we get for $\overline{\psi}_\alpha(x)$ the expansion

$$\overline{\psi}_\alpha(x)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}C_{2k}n^{2k}\right)(l_n^{(\alpha)}\cos nx+p_n^{(\alpha)}\sin nx),$$

since, by (124), we may change the order of summation after replacing in (125) $\varphi_\alpha^{(2k)}(x)$ by the series (126). We have therefore

$$(127)\qquad \overline{\psi}_\alpha(x)=\sum_{n=1}^{\infty}F(n)\,(l_n^{(\alpha)}\cos nx+p_n^{(\alpha)}\sin nx).$$

Thus we may write

$$\gamma_\alpha(x)=\sum_{n=1}^{\infty}(a_n^{(\alpha)}\cos nx+b_n^{(\alpha)}\sin nx),$$

$$a_n^{(\alpha)}=\frac{1}{\pi}\int_0^\alpha\gamma_\alpha(x)\cos nx\,dx,\quad b_n^{(\alpha)}=\frac{1}{\pi}\int_0^\alpha\gamma_\alpha(x)\sin nx\,dx$$

with

$$(128)\qquad a_n^{(\alpha)}=\frac{F(n)}{A_2}\,l_n^{(\alpha)},\ b_n^{(\alpha)}=\frac{F(n)}{A_2}p_n^{(\alpha)}.$$

Since, for $n=n_i$, and $n=0$, $F(n)=0$, we see that

$$a_0=a_{n_i}=b_{n_i}=0(i\geqq 1).$$

Thus the property (3) of the Lemma is proved.

From (123) and from

$$|F(n)|<A_5e^{n^{\sigma+\epsilon}},$$

it follows that (127) may be differentiated, term by term; in other words, $\psi_\alpha(x)$ and $\gamma_\alpha(x)$, are i. d. functions.

We have also

$$(129) \qquad \psi_\alpha^{(k)}(x) = \sum_{n=1}^{\infty} (-1)^n C_{2n} \varphi_\alpha^{(2n+k)}(x),$$

since this series converges uniformly. Indeed, by (119) and (120), a majorant of this series is given by the series

$$A_1 \sum_{n=1}^{\infty} n^{-2n(1/\sigma-\eta_2)} \left[\frac{(2n+k)^{1+\eta_1}}{\alpha.2^{1+\eta_1}} \right]^{2n+k},$$

which converges.

Thus (129) converges uniformly. But since,

$$\varphi_\alpha^{(2n+k)}(0) = \varphi_\alpha^{(2n+k)}(\alpha) = 0,$$

we see that $\bar{\psi}_\alpha^{(k)}(0) = \bar{\psi}_\alpha^{(k)}(\alpha) = 0$, and

$$\gamma_\alpha^{(k)}(0) = \gamma_\alpha^{(k)}(\alpha) = 0.$$

Thus the properties (1) and (2) are satisfied.

It remains to prove the property (4).

Since $\varphi(x)$ is not identically zero in $[0, 1]$ and not negative, there exists an interval $[\beta, \beta_1]$ such that for $\beta \leqq x \leqq \beta_1$, $\beta < \beta_1 \leqq 1$ we have

$$\varphi(x) \geqq d > 0.$$

It follows from this inequality that

$$\varphi_\alpha(x) \geqq d \quad (\alpha\beta \leqq x \leqq \alpha\beta_1).$$

On the other hand it follows from (128) that

$$a a_m^{(\alpha)} + b b_m^{(\alpha)} = \frac{F(m)}{\pi A_2} \int_0^\alpha \varphi_\alpha(x)(a \cos mx + b \sin mx) dx.$$

Since $a^2 + b^2 > 0$, we shall have for x small $(0 < x < \delta)$,

$$|a \cos mx + b \sin mx| > rx,$$

where r is a positive constant. Since, on the other hand, $F(m) \neq 0$, we have for $\alpha < \delta$,

$$|a a_m^{(\alpha)} + b b_m^{(\alpha)}| \geqq \frac{|F(m)|}{\pi A_2} \, d \int_{\alpha\beta}^{\alpha\beta_1} rx dx = \omega\alpha^2.$$

Thus our Lemma is completely proved.

Let us now prove our theorem.

Let $f(x)$ be integrable in $[0, 2\pi]$ and let x_0 be a right-hand mean-value zero of $f(x)$, of exponential order. Let us define the continuation of $f(x)$ by periodicity, with period 2π, possibly changing the value of $f(2\pi)$, in order to have $f(2\pi)=f(0)$, and let us put

$$\varphi(x)=f(x+x_0),\ (0\leqq x\leqq 2\pi).$$

The point $x=0$ is a right-hand mean-value zero for $\varphi(x)$, of the exponential order ρ, and if

$$f(x)\sim\frac{a_0}{2}+\sum (a_n\cos nx+b_n\sin nx),$$

$$\varphi(x)\sim\frac{c_0}{2}+\sum (c_n\cos nx+d_n\sin nx),$$

we have

$$a_n^2+b_n^2=c_n^2+d_n^2;$$

thus the two equalities

$$a_n^2+b_n^2=0,$$
$$c_n^2+d_n^2=0,$$

are satisfied for the same values of n.

If x_0 is a left-hand mean-value zero for $f(x)$, of exponential order ρ, the function

$$\Psi(x)=f(2\pi-x)$$

admits the points $2\pi-x_0$ as a right-hand mean-value zero, of the same exponential order ρ, and if

$$\Psi(x)\sim\frac{l_0}{2}+\sum (l_n\cos nx+m_n\sin nx),$$

we still have

$$l_n^2+m_n^2=a_n^2+b_n^2.$$

Therefore, we can suppose, in our theorem, without diminishing its generality, that $x_0=0$, and that it is a right-hand mean-value zero of $f(x)$, of exponential order ρ.

We have to prove that for every m_i

$$a_{m_i}^2+b_{m_i}^2=0.$$

Suppose then that this is not true for $i=p$. Put

$$m=m_p,\ a=a_{m_p},\ b=b_{m_p}$$

and consider the sequence $\{n_i\}$ of which the terms are

$$m_1,\ m_2,\ \cdots,\ m_{p-1},\ m_{p+1},\ \cdots.$$

The index of convergence of this sequence is obviously also σ. Let us then apply Lemma IX, to this sequence $\{n_i\}$ of integers, to the numbers $m=m_p$, $a=a_{m_p}$, $b=b_{m_p}$ and to a ρ such that

$$\delta>\rho>\frac{\sigma}{1-\sigma}.$$

For small α we have a corresponding family of functions $\gamma_\alpha(x)$ satisfying the properties (1), (2), (3), (4), (5). Let us then introduce the functions $\Gamma_\alpha(x)$ defined, in $[0, 2\pi]$, by continuation of $\gamma_\alpha(x)$:

$$\Gamma_\alpha(x)=\begin{cases}\gamma_\alpha(x) & \text{if } 0\leqq x\leqq\alpha\\ 0 & \text{if } \alpha\leqq x\leqq 2\pi\end{cases}.$$

We have

$$\Gamma_\alpha(x)=\sum_{n=1}^{\infty}(a_n^{(\alpha)}\cos nx+b_n^{(\alpha)}\sin nx),$$

with

$$a_0^{(\alpha)}=a_{n_i}^{(\alpha)}=b_{n_i}^{(\alpha)}=0,$$

$$|a_{m_p}a_{m_p}^{(\alpha)}+b_{m_p}b_{m_p}^{(\alpha)}|>\omega\alpha^2,$$

$$|\Gamma_\alpha(x)|<e^{\alpha^{-\rho}}.$$

By Parseval's formula we have

$$\frac{1}{\pi}\int_0^{2\pi}f(x)\Gamma_\alpha(x)dx=\sum_{n=1}^{\infty}(a_na_n^{(\alpha)}+b_nb_n^{(\alpha)}).^1$$

But if $n\neq m_i$

$$a_n=b_n=0,$$

and, if $n=n_i$,

$$a_{n_i}^{(\alpha)}=b_{n_i}^{(\alpha)}=0,$$

thus, except for $n=m_p$, we have

$$a_na_n^{(\alpha)}+b_nb_n^{(\alpha)}=0.$$

[1]Parseval's formula is true, for instance, if one of the functions is integrable, the other being such that its Fourier series converges uniformly [9]. This is the case for $f(x)$ and $\Gamma_\alpha(x)$.

In other words,

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,\Gamma_\alpha(x)dx = a_{m_p}a_{m_p}^{(\alpha)} + b_{m_p}b_{m_p}^{(\alpha)}.$$

And by Lemma IX

$$\omega\alpha^2 < \frac{1}{\pi}\left|\int_0^{2\pi} f(x)\Gamma_\alpha(x)dx\right| = \frac{1}{\pi}\left|\int_0^{\alpha} f(x)\gamma_\alpha(x)dx\right| < e^{\alpha^{-\rho}}\cdot\frac{1}{\pi}\int_0^{\alpha}|f(x)|\,dx.$$

But, since $x=0$ is a right-hand mean-value zero of $f(x)$, of exponential order δ, to every positive $\epsilon>0$, there corresponds a sequence $\{\alpha_i\}$, such that $\alpha_i \downarrow 0$, and such that

$$\int_0^{\alpha_i}|f(x)|\,dx < e^{-\alpha_i^{-\delta+\epsilon}}.$$

If $a_{m_p}^2 + b_{m_p}^2$ were positive, we should have

$$\pi\omega\alpha_i^2 < e^{\alpha_i^{-\rho}-\alpha_i^{-\delta+\epsilon}},$$

and if $\epsilon<\delta-\rho$, i.e., $\rho-\delta+\epsilon=-\eta<0$, we should have for $\alpha_i \downarrow 0$

$$\pi\omega\alpha_i^2 < e^{-\alpha_i^{-\rho}(\alpha_i^{\rho-\delta+\epsilon}-1)} = e^{-\alpha_i^{-\rho}(\alpha_i^{-\eta}-1)}$$

which is impossible. Thus

$$a_{m_p}^2 + b_{m_p}^2 = 0.$$

And, since this equality is satisfied for every p, our theorem is proved.

This theorem was proved by the author of these lectures in 1932 (**9**). A new proof of the theorem, based on the theory of Fourier transforms, but closely related to the one given, was published later in a common work by Norbert Wiener and the author (**11**), (**21**). Both of us have also proved that, from a certain point of view, the given theorem cannot be improved.

We have proved in fact that, given any $\sigma<1$, it is possible to construct an integrable function $f(x)$, of the form

$$f(x) = \sum_{i=1}^{\infty} a_{n_i}\cos n_i x,\ [0 \leqq x \leqq 2\pi],$$

not identically zero, for which $x=0$ is a right-hand mean-value zero, of exponential order δ, with

$$\delta=\frac{\sigma}{1-\sigma},$$

the index of convergence of $\{n_i\}$ being equal to σ (**11**), (**21**).

An analogous theorem, for a set of values σ, was given in the cited author's paper of 1932.[1]

We shall give now some applications of Theorem XIV, by introducing new quasi-analytic classes (**9**).

The class of all functions $f(x)$ bounded and integrable in $[0, 2\pi]$, and such that

$$(130) \qquad f(x) \sim \sum (a_{n_i} \cos n_i x + b_{n_i} \sin n_i x),$$

the index of convergence of the sequence $\{n_i\}$ being not greater than σ, will be denoted by $C(\sigma)$.

A sequence of positive numbers $\{M_n\}$ being given, we shall denote by $C\{{}^{(\sigma)}_{M_n}\}$ the set of functions belonging to both classes $C(\sigma)$ and $C_{\{M_n\}}$. In other words, $C\{{}^{(\sigma)}_{M_n}\}$ is composed of all functions $f(x)$, i. d. in $[0, 2\pi]$, such that (130) is satisfied, the index of convergence of $\{n_i\}$ being not greater than σ, and such that there exists a constant $k=k(f)$, satisfying in $[0, 2\pi]$ the inequalities

$$|f^{(n)}(x)| < k^n M_n.$$

It is obvious that, if $f_1(x)$ and $f_2(x)$ belong to $C(\sigma)$, their sum and their difference belong also to $C(\sigma)$. Since this is also true for $C_{\{M_n\}}$, then if $f_1(x)$ and $f_2(x)$ belong to $C\{{}^{(\sigma)}_{M_n}\}$, the functions $f_1(x) \pm f_1(x)$ belong also to $C\{{}^{(\sigma)}_{M_n}\}$.

A class $C\{{}^{(\sigma)}_{M_n}\}$ will be said to be quasi-analytic, if the only function $f(x)$ belonging to $C\{{}^{(\sigma)}_{M_n}\}$, such that in $x_0 [0 \leqq x_0 \leqq 2\pi]$ $f^{(n)}(x_0)=0 (n \geqq 0)$, is the function identically equal to zero.

[1]During the printing of this pamphlet, there appeared a paper by Levine and Lifshetz: "Quasi-analytic Functions Represented by Fourier Series," *Recueil Mathematique*, Moscow, T. 9 (51) N. 3 (1941), in which the authors give a profound generalization of the results proved above, their proofs being based on new principles.

In other words, $C_{\{M_n\}}^{(\sigma)}$ is quasi-analytic if, from $f_1(x) \epsilon C_{\{M_n\}}^{(\sigma)}$, $f_2(x) \epsilon C_{\{M_n\}}^{(\sigma)}$ and from

$$f_1^{(n)}(x_0) = f_2^{(n)}(x_0) (n \geqq 0, \ 0 \leqq x_0 \leqq 2\pi),$$

it follows that $f_1(x)$ and $f_2(x)$ are identically equal in $[0, 2\pi]$.

It is obvious that, if $C_{\{M_n\}}$ is quasi-analytic, then $C_{\{M_n\}}^{(\sigma)}$ is quasi-analytic for every $\sigma \leqq 1$. But a class $C_{\{M_n\}}^{(\sigma)}$ with $\sigma < 1$ might be quasi-analytic, the class $C_{\{M_n\}}$ being not quasi-analytic. In order to show this we shall first prove the following theorem.

THEOREM XV. *Let $\{M_n\}$ be a positive sequence such that*

$$(131) \qquad 1 < \varliminf_{n=\infty} \frac{\log M_n}{n \log n} = \mu < \infty.$$

If, in $[a, b]$, $f(x)$ belongs to $C_{\{Mn\}}$, and if

$$f^{(n)}(a) = 0 (n \geqq 0),$$

then a is a right-hand mean-value zero of $f(x)$ of exponential order, at least equal to δ, given by the equality

$$\delta = \frac{1}{\mu - 1}.$$

Indeed there exists a constant k such that in $[a, b]$

$$|f^{(n)}(x)| < k^n M_n.$$

On the other hand, for every $a \leqq x \leqq b$, we have, by Taylor's formula,

$$f(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n,$$

where $a < \xi < b$. Thus *if* $a \leqq x < b$, we have

$$|f(x)| < \frac{k^n}{n!} M_n (x-a)^n (n \geqq 1).$$

Therefore if $a \leqq x \leqq a + \alpha < b$, we have

$$|f(x)| < \frac{k^n}{n!} M_n \alpha^n.$$

But it follows from (131) and from Stirling's formula that

$$\lim \left(\frac{M_n}{n!}\right)^{1/n} = \infty.$$

Thus in $[a, a+\alpha]$ we have

$$|f(x)| \leqq \frac{1}{\underset{n\geqq 1}{\operatorname{Max}}\left(\frac{1}{(\alpha k)^n}\cdot\frac{n!}{M_n}\right)} = \frac{1}{T_N\left(\frac{1}{\alpha k}\right)}, \tag{132}$$

where

$$T_N(r) = \underset{n\geqq 1}{\operatorname{Max}} \frac{r^n}{N_n},$$

the quantities N_n being given by

$$N_n = \frac{M_n}{n!}.$$

Any $\epsilon > 0$ being given, it follows from (131) that, for an infinity of values of n,

$$N_n = \frac{M_n}{n!} < Ae^n n^{(\mu-1+\epsilon)n} = Ae^n n^{(1/\delta+\epsilon)n} = L_n,$$

where A is a constant (depending on ϵ).

Therefore by Lemma III$^{\text{(III)}}$, $T_N(r)$ cannot be smaller than $T_{\mathrm{L}}(r)$, where

$$T_{\mathrm{L}}(r) = \underset{n\geqq 1}{\operatorname{Max}} \frac{r^n}{L_n} = \underset{n\geqq 1}{\operatorname{Max}} \frac{r^n}{Ae^n n^{(1/\delta+\epsilon)n}} > \frac{1}{A}e^{Br^{\delta/(1+\epsilon\delta)}}$$

B being a constant (depending on ϵ), for all large values of r. Thus there exists a sequence $\{r_i\}$, $r_i \uparrow \infty$, such that

$$T_N(r_i) > \frac{1}{A}e^{Br_i^{\delta/(1+\epsilon\delta)}}.$$

And by (132) there exists a sequence $\{\alpha_i\}$, $\alpha_i \downarrow 0$, such that, if $0 \leqq x \leqq \alpha_i$, the following inequalities are satisfied:

$$|f(x)| \leqq \frac{1}{T_N\left(\frac{1}{\alpha_i k}\right)} < Ae^{-B(\alpha_i k)^{-\delta/(1+\epsilon\delta)}} = Ae^{-C\alpha_i^{-\delta/(1+\epsilon\delta)}},$$

where C is a constant.

Therefore, for i large (such that $\alpha_i < b-a$, $A\alpha_i < 1$),

$$\int_a^{a+\alpha_i} |f(x)|\, dx < e^{-C\alpha_i^{-\delta/(1+\epsilon\delta)}}$$

$$-\log\left(\int_a^{a+\alpha_i} |f(x)|\, dx\right) > C\alpha_i^{-\delta/(1+\epsilon\delta)}.$$

And

$$\underline{\lim} \frac{\log\left(-\log\int_a^{a+\alpha}|f(x)|\,dx\right)}{-\log\alpha} \geqq \frac{\delta}{1+\epsilon\delta}.$$

Since this inequality is satisfied for every positive ϵ, our theorem is proved.

From theorems XIV and XV follows immediately:

THEOREM XVI. *The class $C_{\{M_n\}}^{(\sigma)}$ is quasi-analytic, if $\sigma<1$ and if*

$$1<\underline{\lim}\frac{\log M_n}{n\log n}<\frac{1}{\sigma}.$$

This theorem, which was proved by the author in 1932 (**9**) cannot be improved. In other words to every $\sigma<1$ corresponds a sequence $\{M_n\}$ such that

$$\underline{\lim}\frac{\log M_n}{n\log n}=\frac{1}{\sigma},$$

the class $C_{\{M_n\}}^{(\sigma)}$ being not quasi-analytic. This statement was proved by N. Wiener and the present author (**11**), (**21**).

A quantity $\sigma<1$ being given, consider a sequence $\{M_n\}$ such that

$$1<\underline{\lim}\frac{\log M_n}{n\log n}<\frac{1}{\sigma}. \tag{133}$$

The class $C_{\{M_n\}}^{(\sigma)}$ is then quasi-analytic. But the class $C_{\{M_n\}}$ is not quasi-analytic, since, from the first part of this inequality it follows that there exists a constant $\alpha>1$, such that for n large

$$M_n>n^{\alpha n},$$

and we can see immediately that

$$\int_1^\infty \frac{\log T_M(r)}{r^2}\,dr<\infty.$$

Thus the classes $C_{\{M_n\}}^{(\sigma)}$ satisfying (133), which are quasi-analytic, are subclasses of classes $C_{\{M_n\}}$, which are not quasi-analytic. These classes $C_{\{M_n\}}^{(\sigma)}$, are therefore new quasi-analytic classes.

All the quasi-analytic classes we have been concerned with until now, the classes $C_{\{M_n\}}$, $C^*_{\{M_n\}}$, $C^o_{\{M_n\}}$, $C_{\{M_n\}}^{(\sigma)}$, are such that a function which belongs to any one of them is defined in a unique manner if the values of the function and its derivatives are given in one point. Such quasi-analytic classes could be called quasi-analytic-Δ; quasi-analytic-D should be reserved for the classes $C_{\{M_n\}}$, since the first conditions for their quasi-analyticity were given by Denjoy.

If a class C of functions defined on an interval $[a, b]$ is such that from $f_1 \epsilon C$, $f_2 \epsilon C$ it follows that $f_1 \pm f_2$ belong, both, to C, and that the only function which belongs to C and which is zero on a partial interval of $[a, b]$, is the function identically zero (or zero almost everywhere), then the class C should be called quasi-analytic-I. The functions belonging to such a class C have not to be necessarily i. d. S. Bernstein gave conditions for such a quasi-analyticity (**1**).

We shall consider a kind of quasi-analyticity of a larger character (**9**).

Let E be a measurable set of points lying on an interval $[a, b]$. Let x_0 be a point of $[a, b]$, and let α be a quantity such that the point $x_0+\alpha$ belongs also to $[a, b]$; this quantity α may be either positive or negative. We shall denote then by

$$m(E,\ x_0,\ \alpha)$$

the Lebesgue measure of the intersection of E and the interval $[x_0, x_0+\alpha]$.

Let now $\tau(r)$ be a positive function defined for large positive values of r.

We shall say that a point x_0, belonging to $[a, b]$, is a *density point of the set A, with respect to $\tau(r)$*, if

$$\lim_{\overline{\alpha=0}} \frac{m([a, b]-A, x_0, \alpha)}{|\alpha|} \tau\left(\frac{1}{|\alpha|}\right)=0.$$

Intuitively speaking, the greater the rapidity of increase of $\tau(r)$ with r, the greater is the condensation of the points of A in the neighborhood of x_0.

We shall now prove the following theorem (9).

THEOREM XVII. *Let $f(x)$ be a bounded measurable function in the interval $[a, b]$. If $f(x)=0$, at all the points of a set A, for which a point x_0 of $[a, b]$ is a density point with respect to the function e^{r^ρ} ($\rho>0$), then the point x_0 is, for $f(x)$, a mean-value zero (either right-hand or left-hand), of exponential order at least equal to ρ.*

Remark, for the proof, that there exists, by definition of a density point, a sequence $\{\alpha_i\}$, $|\alpha_i| \downarrow 0$, the α_i being either all positive or all negative, such that

$$\lim_{i=\infty} \frac{m([a, b]-A, x_0, \alpha_i)}{\alpha_i} e^{|\alpha_i|^{-\rho}}=0.$$

Let M be such that

$$|f(x)|<M.$$

Supposing that the α_i are all positive (the proof is the same if they are all negative), we shall have for large i,

$$\int_{x_0}^{x_0+\alpha_i} |f(x)|\, dx < M \cdot m([a, b]-A, x_0, \alpha) < M\alpha_i e^{-\alpha_i^{-\rho}},$$

from which it follows easily that

$$\varliminf_{\alpha=+0} \frac{\log\left(-\log \int_{x_0}^{x_0+\alpha} |f(x)|\, dx\right)}{-\log \alpha} \geqq \varliminf_{i=\infty} \frac{\log\left(-\log \int_{x_0}^{x_0+\alpha_i} |f(x)|\, dx\right)}{-\log \alpha_i} \geqq \rho.$$

Thus in this case ($\alpha_i>0$), x_0 is a right-hand mean-value zero for $f(x)$ of exponential order at least equal to ρ. If the α_i were negative we could prove that x_0 is a left-hand mean-value zero for $f(x)$, of exponential order at least equal to ρ.

The theorem is thus proved.

A class C of measurable and bounded functions, in $[a, b]$ such that from $f_1 \epsilon C$, $f_2 \epsilon C$ it follows that $f_1 \pm f_2$ belong, both, to C, will be said to be *quasi-analytic-$E\rho$*, if from

$$f_1(x) = f_2(x),$$

in a set of points having in $[a, b]$ a density point with respect to e^{r^ρ}, it follows that the two functions are equal almost everywhere.

From Theorems XIV and XVII follows:

THEOREM XVIII. *Every class $C(\sigma)$, with $\sigma < 1$, is quasi-analytic $E\rho$, with* $\rho > \dfrac{\sigma}{1-\sigma}$.

Returning to general classes $C_{\{M_n\}}$, we shall mention that, in a recent paper, the author and F. E. Ulrich have given conditions, in order that a function $f(x)$, belonging to $C_{\{M_n\}}$ in $[0, \infty)$, and satisfying the conditions

$$\int_0^\infty |f(x)|\,dx < \infty, \tag{134}$$

$$f^{(\lambda_j)}(0) = 0 \quad (j \geqq 0), \tag{135}$$

where $\{\lambda_j\}$ is a sequence of increasing positive integers, be identically zero.

The two authors have also given conditions in order that any even function belonging to $C^*_{\{M_n\}}$, and satisfying (135), be identically zero. In both cases the only requirement for the sequence $\{\lambda_j\}$ is that

$$\overline{\lim_{j=\infty}} \frac{\lambda_j}{j} < 2.$$

The desired conditions bear, then, on a function, similar to the function $T(r)$, in the classical case, and formed by means of the two sequences $\{M_n\}$ and $\{\lambda_j\}$.